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Differential Equations Invariant under Two-Parameter Lie Groups with Applications to Nonlinear Diffusion

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Differential Equations Invariant under Two-Parameter Lie Groups with Applications to Nonlinear Diffusion

by

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DIFFERENTIAL EQUATIONS INVARIANT UNDER TWO-PARAMETER LIE GROUPS WITH APPLICATIONS TO NONLINEAR DIFFUSION

by

Roy Arthur Axford

ABSTRACT

This report considers two general problems, viz., (1) the determination of the general forms of second-order, linear and nonlinear, ordinary differential equations that are invariant under two-parameter Lie groups, and (2) the exploitation of invariance properties to solve nonlinear differential equations that arise in diffusion phenomena.

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1. INTRODUCTION

The application of the theory of continuous groups of transformations to the solution of ordinary differential equations, partial differential equations, and systems of ordinary and partial differential equations provides a unified approach in the task of obtaining explicit solutions. The theory of continuous groups is developed by Lie[1] and Eisenhart.[2] The application of the theory of continuous groups to the solution of differential equations is given by Lie,[3] Cohen,[4] Dickson,[5] and Engel and Faber.[6] A discussion of the use of infinitesimal transformations in the determination of the Riemann function of second-order, hyperbolic differential operators has been given by Daggit.[7]

The second section of this report shows how the general form of second-order, ordinary differential equations invariant under two-parameter Lie groups in canonical form may be obtained from a knowledge of the invariants and the first and the second differential invariants of these groups.

The third section contains proofs of the invariance of the general forms of 16 second-order, orindary differential equations under 16 two-parameter Lie groups not in canonical form. In each case the type of two-parameter group is identified because this fact is necessary for the reduction of a differential equation invariant under a two-parameter group to its canonical form, which is more easily integrated.

The fourth section contains detailed, exact solutions of nonlinear differential equations that arise in diffusion phenomena. These solutions are obtained by exploiting the fact that the nonlinear diffusion equations are invariant under continuous groups of point transformations. Although the fact of this invariance was arrived at indirectly, rigorous proofs of the invariance properties are given. Particular emphasis is given to establishing conditions that must be satisfied for the existence of solutions of the nonlinear diffusion equations as well as to obtaining their explicit solutions.

2. DIFFERENTIAL EQUATIONS THAT ARE INVARIANT UNDER TWO-PARAMETER LIE GROUPS IN CANONICAL FORM

2.1 Canonical Forms of Two-Parameter Lie Groups

Let

$$U_1f \equiv \xi_1(x,y)\,\frac{\partial f}{\partial x} + \eta_1(x,y)\,\frac{\partial f}{\partial y}, \qquad (2\text{-}1)$$

and

$$U_2f \equiv \xi_2(x,y)\,\frac{\partial f}{\partial x} + \eta_2(x,y)\,\frac{\partial f}{\partial y} \qquad (2\text{-}2)$$

be the symbols of the infinitesimal transformations of two one-parameter, continuous groups of point transformations in two variables. Then, in accordance with Lie's principal theorem,[1,3] these transformations may be regarded as the basis transformations that generate a two-parameter group of point transformations provided that the commutator of U_1 and U_2, viz.,

$$(U_1U_2)f \equiv U_1(U_2f) - U_2(U_1f) \qquad (2\text{-}3a)$$

$$= [U_1(\xi_2) - U_2(\xi_1)] \frac{\partial f}{\partial x}$$

$$+ [U_1(\eta_2) - U_2(\eta_1)] \frac{\partial f}{\partial y}, \qquad (2\text{-}3b)$$

assumes the form,

$$(U_1U_2)f = e_1U_1f + e_2U_2f, \qquad (2\text{-}4)$$

in which e_1 and e_2 are constants.

Two-parameter Lie groups may be classified into four basic types[3] in accordance with the value assumed by the commutator of the basis transformations and whether or not one of the basis transformations may be obtained from the other by multiplying through by an arbitrary function of x and y. That is, the four fundamental types of two-parameter Lie groups satisfy one of the four following sets of relations.

(1) For the first type,

$$(U_1U_2)f = 0, \qquad (2\text{-}5)$$

and

$$\phi_1(x,y)U_1f + \phi_2(x,y)U_2f \neq 0. \qquad (2\text{-}6)$$

(2) For the second type,

$$(U_1U_2)f = 0, \qquad (2\text{-}7)$$

and

$$\phi_1(x,y)U_1f + \phi_2(x,y)U_2f = 0. \qquad (2\text{-}8)$$

(3) For the third type,

$$(U_1U_2)f = U_1f, \qquad (2\text{-}9)$$

and

$$\phi_1(x,y)U_1f + \phi_2(x,y)U_2f \neq 0. \qquad (2\text{-}10)$$

(4) For the fourth type,

$$(U_1U_2)f = U_1f, \qquad (2\text{-}11)$$

and

$$\phi_1(x,y)U_1f + \phi_2(x,y)U_2f = 0. \qquad (2\text{-}12)$$

In the first and second types, $e_1 = 0$ and $e_2 = 0$, whereas $e_1 = 1$ and $e_2 = 0$ in the third and fourth types. In the second and fourth types, U_2f may be obtained from U_1f upon multiplying through by

$$\rho(x,y) = -\frac{\phi_1(x,y)}{\phi_2(x,y)}, \qquad (2\text{-}13)$$

but U_2f may not be obtained from U_1f in the first and third types by multiplying U_1f through by an arbitrary function of x and y.

If the functions $\xi_1(x,y)$, $\xi_2(x,y)$, $\eta_1(x,y)$, and $\eta_2(x,y)$, which appear in the symbols of the infinitesimal transformations of Eqs. (2-1) and (2-2), take on certain elementary forms, then the corresponding two-parameter Lie groups are said to be canonical form. There is a canonical form for each of the four basic types of two-parameter Lie groups, and these four cases will now be discussed.

If $\xi_1(x,y) = 1$, $\eta_1(x,y) = 0$, $\xi_2(x,y) = 0$, and $\eta_2(x,y) = 1$, then

$$U_1f = \frac{\partial f}{\partial x}, \qquad (2\text{-}14)$$

and

$$U_2f = \frac{\partial f}{\partial y} \qquad (2\text{-}15)$$

are the basis transformations of the canonical form of a two-parameter Lie group of the first type, because Eqs. (2-5) and (2-6) are satisfied, as may be verified by evaluating the commutator of the symbols of Eqs. (2-14) and (2-15).

If $\xi_1(x,y) = 0$, $\eta_1(x,y) = 1$, $\xi_2(x,y) = 0$, and $\eta_2(x,y) = x$, then the symbols

$$U_1f = \frac{\partial f}{\partial y}, \qquad (2\text{-}16)$$

and

$$U_2f = x\frac{\partial f}{\partial y} \qquad (2\text{-}17)$$

represent the basis transformations of a two-parameter Lie group of the second type in canonical form.

If $\xi_1(x,y) = 0$, $\eta_1(x,y) = 1$, $\xi_2(x,y) = x$, and $\eta_2(x,y) = y$, then the basis transformations of a two-parameter Lie group of the third type in canonical form are contained in the symbols

$$U_1f = \frac{\partial f}{\partial y}, \qquad (2\text{-}18)$$

and

$$U_2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}. \qquad (2\text{-}19)$$

In a two-parameter Lie group of the fourth type, $\xi_1(x,y) = 0$, $\eta_1(x,y) = 1$, $\xi_2(x,y) = 0$, and $\eta_2(x,y) = y$, and the symbols of the basis transformations of the canonical form are

$$U_1f = \frac{\partial f}{\partial y}, \qquad (2\text{-}20)$$

and

$$U_2f = y\frac{\partial f}{\partial y}. \qquad (2\text{-}21)$$

We turn now to the derivation of the invariants and the first and second differential invariants of the four fundamental types of two-parameter Lie groups in canonical form.

2.2 Invariants and Differential Invariants of Two-Parameter Lie Groups in Canonical Form

In this section the invariants and the first and second differential invariants will be derived for each of the basis transformations of the four basic types of two-parameter Lie groups written in canonical form. Upon comparison of the invariants and the first and second differential invariants of each of the basis transformations of a given two-parameter group in canonical form, we can write the general form of a second-order, ordinary differential equation that is invariant under the canonical form of the given two-parameter group. The four general forms of second-order, ordinary differential equations admitted by the four canonical forms of two-parameter Lie groups will be given in the next section after the invariants and the first and second differential invariants have been obtained.

Let

$$U_i''f = \xi_i(x,y)\frac{\partial f}{\partial x} + \eta_i(x,y)\frac{\partial f}{\partial y} + \eta_i'(x,y,y')\frac{\partial f}{\partial y'} + \eta_i''(x,y,y',y'')\frac{\partial f}{\partial y''}, \qquad (2\text{-}22)$$

wherein $i = 1,2$, be the symbols of the twice-extended one-parameter groups of point transformations generated by the infinitesimal transformations represented by the symbols of Eqs. (2-1) and (2-2). In Eq. (2-22),

$$\eta_i'(x,y,y') \equiv \frac{d\eta_i}{dx}(x,y) - y'\,\frac{d\xi_i}{dx}(x,y), \qquad (2\text{-}23)$$
$$(i = 1,2),$$

and

$$\eta_i''(x,y,y',y'') \equiv \frac{d\eta_i'}{dx}(x,y,y') - y''\,\frac{d\xi_i}{dx}(x,y), \qquad (2\text{-}24)$$
$$(i = 1,2).$$

Then the invariants and the first and second differential invariants of the one-parameter groups generated by U_if for $i = 1,2$ may be found by solving the first-order, linear, partial differential equations obtained by equating the symbols of the corresponding twice-extended groups to zero, i.e., by solving

$$U_i''f = 0, \quad (i = 1,2). \qquad (2\text{-}25)$$

The systems of ordinary differential equations that are equivalent to Eq. (2-25) are

$$\frac{dx}{\xi_i(x,y)} = \frac{dy}{\eta_i(x,y)} = \frac{dy'}{\eta_i'(x,y,y')} = \frac{dy''}{\eta_i''(x,y,y',y'')}. \qquad (2\text{-}26)$$

An arbitrary function of three linearly independent solutions of Eq. (2-26) will be the general solution of Eq. (2-25), and such a function may be interpreted as a second-order, ordinary differential equation admitted by the one-parameter group generated by the infinitesimal transformations with the symbol, U_if. These three linearly independent solutions provide, in fact, an invariant, the first differential invariant, and the second differential invariant of the continuous group with the symbol, U_if.

Now consider the first type of two-parameter Lie group in canonical form with the symbols of Eqs. (2-14) and (2-15). The symbols of the twice-extended basis transformations of this group are found to be

$$U_1''f = \frac{\partial f}{\partial x}, \qquad (2\text{-}27)$$

and

$$U_2''f = \frac{\partial f}{\partial y}, \qquad (2\text{-}28)$$

with Eqs. (2-22) through (2-24). The system of ordinary differential equations that corresponds to $U_1''f = 0$ is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dy'}{0} = \frac{dy''}{0}, \qquad (2\text{-}29)$$

and three linearly independent solutions of this set are

$$u_1(x,y) = y, \qquad (2\text{-}30)$$

$$u_1'(x,y,y') = y', \qquad (2\text{-}31)$$

and

$$u_1''(x,y,y',y'') = y''. \qquad (2\text{-}32)$$

In these last three equations, $u_1(x,y)$ is an invariant, $u_1'(x,y,y')$ is a first differential invariant, and $u_1''(x,y,y',y'')$ is a second differential invariant, all of the one-parameter group of point transformations whose infinitesimal transformation has the symbol of Eq. (2-14). The general form of a second-order, ordinary differential equation admitted by the one-parameter group of point transformations with the symbol of Eq. (2-14) is

$$f(y,y',y'') = 0. \qquad (2\text{-}33)$$

However, this differential equation is not invariant under the two-parameter group of point transformations generated by the two infinitesimal transformations with the symbols of Eqs. (2-14) and (2-15).

The invariants of the second one-parameter group in the two-parameter group defined by Eqs. (2-14) and (2-15) are obtained by equating the right-hand side of Eq. (2-28) to zero, i.e., $U_2''f = 0$, and solving the corresponding system,

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dy'}{0} = \frac{dy''}{0}. \qquad (2\text{-}34)$$

This produces

$$u_2(x,y) = x, \qquad (2\text{-}35)$$

$$u_2'(x,y,y') = y', \qquad (2\text{-}36)$$

and

$$u_2''(x,y,y',y'') = y'' \qquad (2\text{-}37)$$

as an invariant, a first differential invariant, and a second differential invariant, respectively, of the one-parameter group of point transformations generated by the infinitesimal transformation with the symbol of Eq. (2-15). The general form of the second-order, ordinary differential equation admitted by this one-parameter group is

$$f(x,y',y'') = 0. \qquad (2\text{-}38)$$

However, again, the differential equation of Eq. (2-38) is not invariant under the two-parameter Lie group defined by the symbols of Eqs. (2-14) and (2-15). The general form of a second-order, ordinary differential equation admitted by the first type of a two-parameter Lie group in canonical form as defined by the symbols of Eqs. (2-14) and (2-15) will be given in the next section together with those admitted by the canonical forms of the remaining three types of two-parameter Lie groups.

The symbols of the twice-extended groups of point transformations of the basis transformations for the two-parameter Lie group of the second type in canonical form defined by Eqs. (2-16) and (2-17) are

$$U_1''f = \frac{\partial f}{\partial y}, \qquad (2\text{-}39)$$

and

$$U_2''f = x\,\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}. \qquad (2\text{-}40)$$

The invariant and the first and second differential invariants that correspond to Eq. (2-39) are precisely those given in Eqs. (2-35) through (2-37). The system of ordinary differential equations that is equivalent to the partial differential equation, $U_2''f = 0$, from Eq. (2-40) is

$$\frac{dx}{0} = \frac{dy}{x} = \frac{dy'}{1} = \frac{dy''}{0}. \qquad (2\text{-}41)$$

Accordingly, from the first and second members of Eq. (2-41),

$$u_2(x,y) = x \qquad (2\text{-}42)$$

is an invariant, from the second and third members,

$$u_2'(x,y,y') = y' - \frac{y}{x} \qquad (2\text{-}43)$$

is a first differential invariant, and from the second and fourth members,

$$u_2''(x,y,y') = y'' \qquad (2\text{-}44)$$

is a second differential invariant of the one-parameter group of point transformations whose infinitesimal transformation has the symbol of Eq. (2-17).

In the case of the third type of two-parameter Lie group in canonical form, the two one-parameter groups of basis transformations contained in Eqs. (2-18) and (2-19) have twice-extended groups that are generated by the symbols

$$U_1''f = \frac{\partial f}{\partial y}, \qquad (2\text{-}45)$$

and

$$U_2''f = x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y} - y''\,\frac{\partial f}{\partial y''}, \qquad (2\text{-}46)$$

respectively. The invariant and first and second differential invariants obtained from Eq. (2-45) have been given in Eqs. (2-35) through (2-37). The first-order, linear partial differential equation written from Eq. (2-46) has the following corresponding set of four first-order, ordinary differential equations

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dy'}{0} = \frac{dy''}{-y''}. \qquad (2\text{-}47)$$

The first two members of this set, viz.,

$$\frac{dx}{x} = \frac{dy}{y}, \qquad (2\text{-}48)$$

integrate out to yield

$$u_2(x,y) = \frac{y}{x} \qquad (2\text{-}49)$$

as an invariant of the one-parameter group of point transformations whose infinitesimal transformation symbol is that of Eq. (2-19). The first and third members of Eq. (2-47) produce the first differential invariant,

$$u_2'(x,y,y') = y', \qquad (2\text{-}50)$$

of the same group. A second differential invariant of this group comes out of the first and fourth members of Eq. (2-47) in the form

$$u_2''(x,y,y',y'') = xy''. \qquad (2\text{-}51)$$

The symbols of the twice-extended groups of point transformations of the two basis transformations of the fourth type of two-parameter Lie group in canonical form as given in Eqs. (2-20) and (2-21) are

$$U_1''f = \frac{\partial f}{\partial y}, \qquad (2\text{-}52)$$

and

$$U_2''f = y\,\frac{\partial f}{\partial y} + y'\,\frac{\partial f}{\partial y'} + y''\,\frac{\partial f}{\partial y''}. \qquad (2\text{-}53)$$

Again the invariant and the first and second differential invariants that arise from Eq. (2-52) are those of Eqs. (2-35) through (2-37). The first-order, partial differential equation, $U_2''f = 0$, from Eq. (2-53) is equivalent to the set,

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dy'}{y'} = \frac{dy''}{y''}, \qquad (2\text{-}54)$$

of ordinary differential equations. From the first and second members of this set, we obtain the invariant,

$$u_2(x,y) = x, \qquad (2\text{-}55)$$

from the second and third members, the first differential invariant,

$$u_2'(x,y,y') = \frac{y'}{y}, \qquad (2\text{-}56)$$

and from the second and fourth members, the second differential invariant,

$$u_2''(x,y,y',y'') = \frac{y''}{y}, \qquad (2\text{-}57)$$

all of the one-parameter group of point transformations with the infinitesimal transformation symbol given in Eq. (2-21).

The principal results of this section have been summarized in Table I. In this table the invariants, first differential invariants, and second differential invariants are listed for each of the two basis transformations of the four fundamental types of two-parameter Lie groups written in their canonical forms. The general form of a second-order, ordinary differential equation that is invariant under the group of one-parameter point transformations generated by the first basis transformation of a two-parameter Lie group of the first type in canonical form has been given in Eq. (2-33), and that invariant under the group of the second basis transformation of this same two-parameter group, in Eq. (2-38). The corresponding results for the basis transformations of the two-parameter Lie groups of the second, third, and fourth types in canonical form are as follows.

(a) For the second type,

$$U_1 f = \frac{\partial f}{\partial y} \qquad (2\text{-}58)$$

admits

$$f(x,y',y'') = 0, \qquad (2\text{-}59)$$

and

$$U_2 f = x\,\frac{\partial f}{\partial y} \qquad (2\text{-}60)$$

admits

$$f(x,y'-\frac{y}{x},y'') = 0. \qquad (2\text{-}61)$$

(b) For the third type, $U_1 f = \frac{\partial f}{\partial y}$ admits Eq. (2-59), whereas

Table I. INVARIANTS AND DIFFERENTIAL INVARIANTS OF THE POINT TRANSFORMATIONS OF THE FOUR BASIC TYPES OF TWO-PARAMETER LIE GROUPS IN CANONICAL FORM

Basic Types of Two-Parameter Lie Groups	Invariant	First Differential Invariant	Second Differential Invariant
First Type			
$U_1 f = \frac{\partial f}{\partial x}$	$u_1(x,y) = y$	$u_1'(x,y,y') = y'$	$u_1''(x,y,y',y'') = y''$
$U_2 f = \frac{\partial f}{\partial y}$	$u_2(x,y) = x$	$u_2'(x,y,y') = y'$	$u_2''(x,y,y',y'') = y''$
Second Type			
$U_1 f = \frac{\partial f}{\partial y}$	$u_1(x,y) = x$	$u_1'(x,y,y') = y'$	$u_1''(x,y,y',y'') = y''$
$U_2 f = x\,\frac{\partial f}{\partial y}$	$u_2(x,y) = x$	$u_2'(x,y,y') = y' - \frac{y}{x}$	$u_2''(x,y,y',y'') = y''$
Third Type			
$U_1 f = \frac{\partial f}{\partial y}$	$u_1(x,y) = x$	$u_1'(x,y,y') = y'$	$u_1''(x,y,y',y'') = y''$
$U_2 f = x\,\frac{\partial f}{\partial x} + y\,\frac{\partial f}{\partial y}$	$u_2(x,y) = \frac{y}{x}$	$u_2'(x,y,y') = y'$	$u_2''(x,y,y',y'') = xy''$
Fourth Type			
$U_1 f = \frac{\partial f}{\partial y}$	$u_1(x,y) = x$	$u_1'(x,y,y') = y'$	$u_1''(x,y,y',y'') = y''$
$U_2 f = y\,\frac{\partial f}{\partial y}$	$u_2(x,y) = x$	$u_2'(x,y,y') = \frac{y'}{y}$	$u_2''(x,y,y',y'') = \frac{y''}{y}$

$$U_2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \qquad (2\text{-}62)$$

admits

$$f(\frac{y}{x}, y', xy'') = 0. \qquad (2\text{-}63)$$

(c) For the fourth type, $U_1f = \frac{\partial f}{\partial y}$ admits Eq. (2-59), and

$$f(x, \frac{y'}{y}, \frac{y''}{y}) = 0 \qquad (2\text{-}64)$$

is invariant under

$$U_2f = y\frac{\partial f}{\partial y}. \qquad (2\text{-}65)$$

The fact being considered here is that the general form of the second-order, ordinary differential equation that is invariant under each of the one-parameter groups of point transformations generated by each of the basis transformations of one of the four fundamental types of two-parameter Lie groups written in canonical form may be obtained quite directly, as has been done above, as an arbitrary function of an invariant, a first differential invariant, and a second differential invariant of the corresponding one-parameter group. However, the general forms of the second-order differential equations so obtained are not invariant under the two-parameter groups of point transformations. The general forms of second-order, ordinary differential equations, which may be either linear or nonlinear, and that are invariant under one of the four fundamental two-parameter Lie groups in canonical form, may be obtained by inspection of the invariants and the first and second differential invariants listed in Table I.

2.3 General Forms of Second-Order, Ordinary Differential Equations That Are Invariant under the Four Canonical Forms of Two-Parameter Lie Groups

Although the four general forms of second-order, ordinary differential equations that are invariant under the four fundamental types of two-parameter Lie groups in canonical form may be obtained by inspection, the actual proof of this invariance will be made on the basis of the invariance of the first-order, linear partial differential equation, which is equivalent to the second-order, ordinary differential equation. If a second-order, ordinary differential equation is written in the form

$$y'' = F(x,y,y'), \qquad (2\text{-}66)$$

in which the right-hand side is to be regarded as an arbitrary function of the indicated arguments, then this equation is equivalent to the linear, first-order partial differential equation,

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + F(x,y,y')\frac{\partial f}{\partial y'} = 0, \qquad (2\text{-}67)$$

in three variables. This fact may be established directly by writing Eq. (2-66) as a set of two first-order, ordinary differential equations and noting that Eq. (2-67) is equivalent to this set. The general solution of Eq. (2-67) is an arbitrary function of two linearly independent solutions of the equivalent set of two first-order differential equations. Each of these two linearly independent solutions will, in general, be a function of the three variables, x, y, y', and, if the derivative, y', is eliminated between them, then the general solution of Eq. (2-66) is obtained.

Suppose that Eq. (2-66) admits a one-parameter group of point transformations in two variables whose infinitesimal transformation has the symbol

$$Uf = \xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y}. \qquad (2\text{-}68)$$

Then the equivalent partial differential equation contained in Eq. (2-67) will be invariant under the once-extended group of point transformations generated by the symbol

$$U'f = \xi(x,y)\frac{\partial f}{\partial x} + \eta(x,y)\frac{\partial f}{\partial y} + \left[\frac{d\eta}{dx}(x,y) - y'\frac{d\xi}{dx}(x,y)\right]\frac{\partial f}{\partial y'}. \qquad (2\text{-}69)$$

To determine whether or not Eq. (2-67) admits a given one-parameter group, use[3] may be made of the fact that the commutator between the differential operator, A and U', assumes the form

$$(U'A)f = \lambda(x,y,y')Af, \qquad (2\text{-}70)$$

wherein $\lambda(x,y,y')$ is an arbitrary function of its arguments, when $Af = 0$ admits Uf. Furthermore, if the partial differential equation of Eq. (2-67) is invariant under a one-parameter group of point transformations, the second-order, ordinary differential equation of Eq. (2-66), to which it is equivalent, is also invariant under this same group. Also, if a first-order, linear partial differential equation and its corresponding second-order, ordinary differential equation are going to be invariant under a two-parameter Lie group, then Eq. (2-70) must be satisfied when used with the once-extended symbols of both of the basis transformations of this group. The preceding facts will be used to establish the invariance of second-order, ordinary differential equations under two-parameter Lie groups.

Now consider the invariant and the first and second differential invariants of the basis transformations of the canonical form of a two-parameter Lie group of the first type as given in Table I. By inspection it may be asserted that

$$y'' = F(y') \tag{2-71}$$

is the general form of a second-order, ordinary differential equation admitted by the first type of two-parameter Lie group in canonical form. To prove this assertion, we may first note that

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + F(y')\frac{\partial f}{\partial y'} = 0, \tag{2-72}$$

in which $F(y')$ is an arbitrary function of the derivative, is the linear, first-order partial differential equation equivalent to Eq. (2-71). The symbols of the once-extended groups generated by the basis transformations of the canonical form of the first type of two-parameter Lie group are

$$U_1'f = \frac{\partial f}{\partial x}, \tag{2-73}$$

and

$$U_2'f = \frac{\partial f}{\partial y}. \tag{2-74}$$

With the operators appearing in Eqs. (2-72) and (2-73) we have

$$(U_1'A)f = 0, \tag{2-75}$$

and with those of Eqs. (2-72) and (2-74),

$$(U_2'A)f = 0. \tag{2-76}$$

These last two relations complete the proof that Eq. (2-71) is the general form of a second-order, ordinary differential equation that is invariant under the canonical form of the first type of two-parameter Lie group in two variables.

For the second type of two-parameter Lie group in canonical form

$$y'' = F(x), \tag{2-77}$$

in which $F(x)$ is an arbitrary function of its argument, is the general form of the second-order, ordinary differential equation admitted by this group. The corresponding first-order partial differential equation is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + F(x)\frac{\partial f}{\partial y'} = 0, \tag{2-78}$$

and the symbols of the once-extended basis transformation groups are

$$U_1'f = \frac{\partial f}{\partial y}, \tag{2-79}$$

and

$$U_2'f = x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}. \tag{2-80}$$

Because the commutators formed from Eqs. (2-78) through (2-80) are

$$(U_1'A)f = 0, \tag{2-81}$$

and

$$(U_2'A)f = 0, \tag{2-82}$$

the invariance of Eq. (2-77) under the canonical form of the second type of two-parameter Lie group is established.

By inspection from Table I, the general form of a second-order, ordinary differential equation admitted by the third type of two-parameter Lie group in canonical form is

$$xy'' = F(y'), \tag{2-83}$$

which is equivalent to the first-order partial differential equation

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{F(y')}{x}\frac{\partial f}{\partial y'} = 0. \tag{2-84}$$

The symbols of the groups of the once-extended basis transformations are

$$U_1'f = \frac{\partial f}{\partial y}, \tag{2-85}$$

and

$$U_2'f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}. \tag{2-86}$$

From Eqs. (2-84) and (2-85),

$$(U_1'A)f = 0, \tag{2-87}$$

and from Eqs. (2-84) and (2-86),

$$(U_2'A)f = 0, \tag{2-88}$$

which proves the invariance of Eq. (2-83) under the third type of two-parameter Lie group in canonical form.

The canonical form of the fourth type of two-parameter Lie group admits

$$y'' = F(x)y'. \tag{2-89}$$

The corresponding first-order partial differential equation is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y'F(x)\frac{\partial f}{\partial y'} = 0, \tag{2-90}$$

and the symbols of the once-extended groups of the two basis transformations for this case are

$$U_1'f = \frac{\partial f}{\partial y}, \tag{2-91}$$

and

$$U_2'f = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'}. \tag{2-92}$$

The values of the commutators developed out of Eqs. (2-90) through (2-92) are

$$(U_1'A)f = 0, \tag{2-93}$$

and

$$(U_2'A)f = 0, \qquad (2\text{-}94)$$

which completes the proof that Eq. (2-89) is invariant under the canonical form of the fourth type of two-parameter Lie group.

The principal results of this section have been summarized in Table II. The general forms of the four second-order, ordinary differential equations that are admitted by the canonical forms of the four fundamental types of two-parameter Lie groups of point transformations are seen to be relatively simple and easy to integrate. Consequently, the quadrature question for these four forms need not be discussed further here.

Table II. GENERAL FORMS OF SECOND-ORDER, ORDINARY DIFFERENTIAL EQUATIONS THAT ARE INVARIANT UNDER THE FOUR CANONICAL FORMS OF TWO-PARAMETER LIE GROUPS

Type of Two-Parameter Lie Group	Second-Order, Ordinary Differential Equation Invariant under the Group
First Type: $U_1f = \frac{\partial f}{\partial x}$, $U_2f = \frac{\partial f}{\partial y}$	$y'' = F(y')$
Second Type: $U_1f = \frac{\partial f}{\partial y}$, $U_2f = x\frac{\partial f}{\partial y}$	$y'' = F(x)$
Third Type: $U_1f = \frac{\partial f}{\partial y}$, $U_2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$	$xy'' = F(y')$
Fourth Type: $U_1f = \frac{\partial f}{\partial y}$, $U_2f = y\frac{\partial f}{\partial y}$	$y'' = y'F(x)$

We now turn to the determination of classes of second-order, ordinary differential equations that are admitted by given two-parameter Lie groups that are not, however, in canonical form as was the case in Sec. 2.3.

3. SECOND-ORDER DIFFERENTIAL EQUATIONS INVARIANT UNDER TWO-PARAMETER LIE GROUPS NOT IN CANONICAL FORM

3.1 Second-Order Differential Equations That Are Admitted by Two-Parameter Lie Groups of the First Type

Consider the pair of two one-parameter groups of point transformations whose infinitesimal transformations are represented by the symbols

$$U_1f = \frac{\partial f}{\partial y}, \qquad (3\text{-}1)$$

and

$$U_2f = x\frac{\partial f}{\partial x}. \qquad (3\text{-}2)$$

This pair comprises, in fact, a set of basis transformations for a two-parameter group of point transformations of the first type because the commutator obtained from Eqs. (3-1) and (3-2) is

$$(U_1U_2)f = 0, \qquad (3\text{-}3)$$

and, also, because

$$U_2f \neq \rho(x,y)U_1f. \qquad (3\text{-}4)$$

The second-order, ordinary differential equation

$$x^2y'' = F(xy'), \qquad (3\text{-}5)$$

in which F is an arbitrary function of the indicated argument, xy', is invariant under the two-parameter Lie group whose basis transformations are represented by the symbols of Eqs. (3-1) and (3-2). This fact may be established in the following manner.

The first-order, linear partial differential equation that corresponds to Eq. (3-5) is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{F(xy')}{x^2}\frac{\partial f}{\partial y'} = 0. \qquad (3\text{-}6)$$

The symbols of the once-extended groups of basis transformations found from Eqs. (3-1) and (3-2) are

$$U_1'f = \frac{\partial f}{\partial y}, \qquad (3\text{-}7)$$

and

$$U_2'f = x\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'}. \qquad (3\text{-}8)$$

From Eqs. (3-6) and (3-7), we obtain the commutator

$$(U_1'A)f = 0, \qquad (3\text{-}9)$$

and from Eqs. (3-6) and (3-8), the commutator

$$(U_2'A)f = U_2'(Af) - A(U_2'f), \qquad (3\text{-}10a)$$

$$= \left[U_2'(1) - A(x)\right]\frac{\partial f}{\partial x} + U_2'(y')\frac{\partial f}{\partial y} + \left\{U_2'\left[\frac{F(xy')}{x^2}\right] + A(y')\right\}\frac{\partial f}{\partial y'}, \qquad (3\text{-}10b)$$

$$= -\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'} + \left\{x\left[\frac{-2}{x^3}F(xy') + \frac{y'}{x^2}F'(xy')\right] - \frac{y'}{x}F'(xy') + \frac{F(xy')}{x^2}\right\}\frac{\partial f}{\partial y'}, \qquad (3\text{-}10c)$$

$$= -\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'} - \frac{F(xy')}{x^2}\frac{\partial f}{\partial y'}. \qquad (3\text{-}10d)$$

Therefore,

$$(U_2'A)f = -Af, \qquad (3\text{-}11)$$

and Eqs. (3-9) and (3-11) establish the invariance of Eq. (3-6) and, hence, that of Eq. (3-5) under the two-parameter Lie group of the first type with the basis transformations with the symbols of Eqs. (3-1) and (3-2).

The second-order differential equation,

$$y'' = yF\frac{y'}{y}, \qquad (3\text{-}12)$$

wherein $F(y'/y)$ is an arbitrary function of its argument, is admitted by the two-parameter group with the symbols,

$$U_1f = \frac{\partial f}{\partial x}, \qquad (3\text{-}13)$$

and

$$U_2f = y\,\frac{\partial f}{\partial y}. \qquad (3\text{-}14)$$

The basis transformations of Eqs. (3-13) and (3-14) define a two-parameter group of the first type because they satisfy Eqs. (3-3) and (3-4). The first-order partial differential equation equivalent to Eq. (3-12) is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + yF\left(\frac{y'}{y}\right)\frac{\partial f}{\partial y'} = 0, \qquad (3\text{-}15)$$

and the once-extended groups of basis transformations for this case are represented by the symbols,

$$U_1'f = \frac{\partial f}{\partial x}, \qquad (3\text{-}16)$$

and

$$U_2'f = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'}. \qquad (3\text{-}17)$$

With Eqs. (3-15) through (3-17) the commutators work out to be

$$(U_1'A)f = 0, \qquad (3\text{-}18)$$

and

$$(U_2'A)f = 0, \qquad (3\text{-}19)$$

so that Eqs. (3-12) and (3-15) are invariant under the two-parameter group with the basis transformations of Eqs. (3-13) and (3-14).

We next consider the two-parameter group with the basis transformations,

$$U_1f = x\frac{\partial f}{\partial y}, \qquad (3\text{-}20)$$

and

$$U_2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}. \qquad (3\text{-}21)$$

These last two relations satisfy Eqs. (3-3) and (3-4) and, thereby, comprise a two-parameter group of the first type. The second-order differential equation admitted by this group is

$$xy'' = F\left(y' - \frac{y}{x}\right), \qquad (3\text{-}22)$$

in which F is an arbitrary function of the indicated argument. This follows from the facts that

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{1}{x}F\left(y' - \frac{y}{x}\right)\frac{\partial f}{\partial y'} = 0 \qquad (3\text{-}23)$$

is the equivalent partial differential equation, that

$$U_1'f = x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}, \qquad (3\text{-}24)$$

and

$$U_2'f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \qquad (3\text{-}25)$$

are the symbols of the once-extended basis transformations, and that the commutators formed from Eqs. (3-23) through (3-25) are

$$(U_1'A)f = 0, \qquad (3\text{-}26)$$

and

$$(U_2'A)f = -\,Af. \qquad (3\text{-}27)$$

A fourth example of a two-parameter Lie group of the first type is that group with the basis transformations

$$U_1f = x\frac{\partial f}{\partial x}, \qquad (3\text{-}28)$$

and

$$U_2f = y\frac{\partial f}{\partial y}, \qquad (3\text{-}29)$$

which satisfy Eqs. (3-3) and (3-4). This group admits the second-order differential equation

$$x^2y'' = yF\left(\frac{xy'}{y}\right), \qquad (3\text{-}30)$$

with the equivalent partial differential equation,

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{y}{x^2}F\left(\frac{xy'}{y}\right)\frac{\partial f}{\partial y'} = 0, \qquad (3\text{-}31)$$

as follows from the symbols of the once-extended basis transformations, viz.,

$$U_1'f = x\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'}, \qquad (3\text{-}32)$$

and

$$U_2'f = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'}, \qquad (3\text{-}33)$$

and the commutators,

$$(U_1'A)f = -Af, \qquad (3\text{-}34)$$

and

$$(U_2'A)f = 0, \qquad (3\text{-}35)$$

which come out of Eqs. (3-31) through (3-33).

3.2 Differential Equations Invariant under Two-Parameter Lie Groups of the Second Type

The two-parameter group of point transformations which is represented by the symbols,

$$U_1f = x\frac{\partial f}{\partial y}, \qquad (3\text{-}36)$$

and

$$U_2f = x^2\,\frac{\partial f}{\partial y}, \qquad (3\text{-}37)$$

is of the second type because

$$(U_1U_2)f = 0, \qquad (3\text{-}38)$$

and

$$U_2f = \rho(x,y)U_1f \qquad (3\text{-}39)$$

with $\rho(x,y) = x$. This group admits the second-order, ordinary differential equation,

$$x^2y'' - 2xy' + 2y = F(x), \qquad (3\text{-}40)$$

with the equivalent first-order partial differential equation,

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \left[\frac{2}{x}y' - \frac{2}{x^2}y + \frac{F(x)}{x^2}\right]\frac{\partial f}{\partial y'} = 0. \qquad (3\text{-}41)$$

The symbols of the once-extended basis transformations are

$$U_1'f = x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}, \qquad (3\text{-}42)$$

and

$$U_2'f = x^2\frac{\partial f}{\partial y} + 2x\frac{\partial f}{\partial y'}. \qquad (3\text{-}43)$$

The invariance of Eqs. (3-40) and (3-41) under this group is a consequence of the values of the commutators that arise from Eqs. (3-41) through (3-43), viz.,

$$(U_1'A)f = 0, \qquad (3\text{-}44)$$

and

$$(U_2'A)f = 0. \qquad (3\text{-}45)$$

The second-order differential equation,

$$yy'' = (y')^2 + F(x)y^2, \qquad (3\text{-}46)$$

is invariant under the two-parameter group whose basis transformation symbols are

$$U_1f = xy\frac{\partial f}{\partial y}, \qquad (3\text{-}47)$$

and

$$U_2f = y\frac{\partial f}{\partial y}. \qquad (3\text{-}48)$$

This group is of the second type, since Eqs. (3-47) and (3-48) satisfy Eq. (3-38) and Eq. (3-39) with $\rho(x,y) = 1/x$. The once-extended groups of basis transformations corresponding to Eqs. (3-47) and (3-48) are represented by the symbols,

$$U_1'f = xy\frac{\partial f}{\partial y} + (y + xy')\frac{\partial f}{\partial y'}, \qquad (3\text{-}49)$$

and

$$U_2'f = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'}. \qquad (3\text{-}50)$$

The first-order partial differential equation equivalent to Eq. (3-46) is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \left[\frac{(y')^2}{y} + yF(x)\right]\frac{\partial f}{\partial y'} = 0, \qquad (3\text{-}51)$$

and the commutators which come out of Eqs. (3-49) through (3-51) are

$$(U_1'A)f = 0, \qquad (3\text{-}52)$$

and

$$(U_2'A)f = 0. \qquad (3\text{-}53)$$

The two-parameter group represented by

$$U_1f = \frac{1}{y}\frac{\partial f}{\partial y}, \qquad (3\text{-}54)$$

and

$$U_2f = \frac{x}{y}\frac{\partial f}{\partial y}, \qquad (3\text{-}55)$$

is of the second type. It admits the second-order differential equation,

$$yy'' + (y')^2 = F(x), \qquad (3\text{-}56)$$

equivalent to

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \left[\frac{F(x)}{y} - \frac{(y')^2}{y}\right]\frac{\partial f}{\partial y'} = 0. \qquad (3\text{-}57)$$

The symbols of the once-extended groups of basis transformations are

$$U_1'f = \frac{1}{y}\frac{\partial f}{\partial y} - \frac{y'}{y^2}\frac{\partial f}{\partial y'}, \qquad (3\text{-}58)$$

and

$$U_2'f = \frac{x}{y}\frac{\partial f}{\partial y} + \left[\frac{1}{y} - \frac{xy'}{y^2}\right]\frac{\partial f}{\partial y'}. \qquad (3\text{-}59)$$

Evaluating the relevant commutators with Eqs. (3-57) through (3-59) produces

$$(U_1'A)f = 0, \qquad (3\text{-}60)$$

and

$$(U_2'A)f = 0, \qquad (3\text{-}61)$$

which establishes the invariance of Eq. (3-56) under the group of the symbols of Eqs. (3-54) and (3-55).

The nonlinear, second-order, ordinary differential equation

$$yy'' - 2(y')^2 - \frac{2}{x}yy' - \frac{2}{x^2}y^2 = F(x)y^3, \qquad (3\text{-}62)$$

is invariant under the two-parameter Lie group represented by the basis transformation symbols

$$U_1f = xy^2\frac{\partial f}{\partial y}, \qquad (3\text{-}63)$$

and

$$U_2f = x^2y^2\frac{\partial f}{\partial y}. \qquad (3\text{-}64)$$

This group is of the second type, and the symbols of its once-extended basis transformations are

$$U_1'f = xy^2\frac{\partial f}{\partial y} + (y^2 + 2xyy')\frac{\partial f}{\partial y'}, \qquad (3\text{-}65)$$

and

$$U_2'f = x^2y^2\frac{\partial f}{\partial y} + (2xy^2 + 2x^2yy')\frac{\partial f}{\partial y'}. \qquad (3\text{-}66)$$

The commutators formed with Eqs. (3-65) and (3-66) and the partial differential equation equivalent to

Eq. (3-62), viz.,

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \left[\frac{2}{y}(y')^2 + \frac{2}{x}y' + \frac{2}{x^2}y + F(x)y^2\right]\frac{\partial f}{\partial y'} = 0, \qquad (3\text{-}67)$$

assume the values

$$(U_1'A)f = 0, \qquad (3\text{-}68)$$

and

$$(U_2'A)f = 0 \qquad (3\text{-}69)$$

to complete the invariance proof.

3.3 Differential Equations Invariant under Two-Parameter Lie Groups of the Third Type

The basis transformations with the symbols

$$U_1f = \frac{\partial f}{\partial x}, \qquad (3\text{-}70)$$

and

$$U_2f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \qquad (3\text{-}71)$$

generate a two-parameter Lie group of the third type as they satisfy the relations

$$(U_1U_2)f = U_1f, \qquad (3\text{-}72)$$

and

$$U_2f \neq \rho(x,y)U_1f. \qquad (3\text{-}73)$$

This group admits the second-order differential equation

$$yy'' = F(y') \qquad (3\text{-}74)$$

with the equivalent first-order partial differential equation

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{F(y')}{y}\frac{\partial f}{\partial y'} = 0. \qquad (3\text{-}75)$$

The once-extended groups that arise from Eqs. (3-70) and (3-71) are represented by the symbols

$$U_1'f = \frac{\partial f}{\partial x}, \qquad (3\text{-}76)$$

and

$$U_2'f = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}. \qquad (3\text{-}77)$$

The invariance of both Eq. (3-74) and (3-75) under the two-parameter group represented by Eqs. (3-70) and (3-71) is a consequence of the fact that the commutators formed with Eqs. (3-75) through (3-77) assume the values

$$(U_1'A)f = 0, \qquad (3\text{-}78)$$

and

$$(U_2'A)f = -Af. \qquad (3\text{-}79)$$

The second-order differential equation

$$y'' = x^{n-2}F(x^{1-n}y') \qquad (3\text{-}80)$$

is invariant under the two-parameter group generated by the symbols

$$U_1f = \frac{\partial f}{\partial y}, \qquad (3\text{-}81)$$

and

$$U_2f = x\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y}. \qquad (3\text{-}82)$$

To show that this group is of the third type of two-parameter Lie group, we introduce the basis transformation symbols

$$V_1f \equiv U_1f = \frac{\partial f}{\partial y}, \qquad (3\text{-}83)$$

and

$$V_2f \equiv \frac{1}{n}U_2f = \frac{x}{n}\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \qquad (3\text{-}84)$$

and observe that

$$(V_1V_2)f = V_1f \qquad (3\text{-}85)$$

together with

$$V_2f \neq \rho(x,y)V_1f. \qquad (3\text{-}86)$$

The linear, first-order partial differential equation that corresponds to Eq. (3-80) is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + x^{n-2}F(x^{1-n}y')\frac{\partial f}{\partial y'} = 0, \qquad (3\text{-}87)$$

and the once-extended groups that come out of Eqs. (3-81) and (3-82) are represented by the symbols,

$$U_1'f = \frac{\partial f}{\partial y}, \qquad (3\text{-}88)$$

and

$$U_2'f = x\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y} + (n-1)y'\frac{\partial f}{\partial y'}. \qquad (3\text{-}89)$$

Because the commutators constructed from Eqs. (3-87) through (3-89) are

$$(U_1'A)f = 0, \qquad (3\text{-}90)$$

and

$$(U_2'A)f = \left[U_2'(1) - A(x)\right]\frac{\partial f}{\partial x} + \left[U_2'(y') - A(ny)\right]\frac{\partial f}{\partial y} + \left\{U_2'\left[x^{n-2}F(x^{1-n}y')\right] - A\left[(n-1)y'\right]\right\}\frac{\partial f}{\partial y'}, \qquad (3\text{-}91a)$$

$$= -\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + \left\{x\left[(n-2)x^{n-3}F(x^{1-n}y') + x^{n-2}x^{1-n-1}(1-n)y'F'(x^{1-n}y')\right] + (n-1)y'x^{n-2}x^{1-n}F'(x^{1-n}y') - (n-1)x^{n-2}F(x^{1-n}y')\right\}\frac{\partial f}{\partial y'}, \qquad (3\text{-}91b)$$

$$= -\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} - x^{n-2}F(x^{1-n}y')\frac{\partial f}{\partial y'}, \qquad (3\text{-}91c)$$

$$= -Af, \qquad (3\text{-}91d)$$

the invariance of Eqs. (3-80) and (3-87) under the two-parameter group with the symbols of Eqs. (3-81) and (3-82) is proved.

The two-parameter group generated by the two infinitesimal transformations with the symbols

$$U_1f = x\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y}, \qquad (3\text{-}92)$$

and

$$U_2f = x\frac{\partial f}{\partial y} \qquad (3\text{-}93)$$

is of the third type. This fact follows by selecting the basis transformations as

$$V_1f \equiv U_2f = x\frac{\partial f}{\partial y}, \qquad (3\text{-}94)$$

and

$$V_2f = \frac{x}{n-1}\frac{\partial f}{\partial x} + \frac{n}{n-1}y\frac{\partial f}{\partial y}, \qquad (3\text{-}95)$$

and by noting that the commutator constructed with Eqs. (3-94) and (3-95) is

$$(V_1V_2)f = V_1f, \qquad (3\text{-}96)$$

and that

$$V_2f \neq \rho(x,y)V_1f. \qquad (3\text{-}97)$$

The second-order differential equation,

$$y'' = x^{n-2}F\left(\frac{y'}{x^{n-1}} - \frac{y}{x^n}\right), \qquad (3\text{-}98)$$

together with its equivalent first-order partial differential equation, viz.,

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + x^{n-2}F\left(\frac{y'}{x^{n-1}} - \frac{y}{x^n}\right)\frac{\partial f}{\partial y'} = 0 \qquad (3\text{-}99)$$

are admitted by the group of point transformations defined by Eqs. (3-92) and (3-93), because the corresponding once-extended groups are generated by the symbols

$$U_1'f = x\frac{\partial f}{\partial x} + n\frac{\partial f}{\partial y} + (n-1)y'\frac{\partial f}{\partial y'}, \qquad (3\text{-}100)$$

and

$$U_2'f = x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}, \qquad (3\text{-}101)$$

and because the commutators that come out of Eqs. (3-99) through (3-101) assume the value

$$(U_1'A)f = -Af, \qquad (3\text{-}102)$$

and

$$(U_2'A)f = 0. \qquad (3\text{-}103)$$

The symbols

$$U_1f = x\frac{\partial f}{\partial x}, \qquad (3\text{-}104)$$

and

$$U_2f = x\frac{\partial f}{\partial y} \qquad (3\text{-}105)$$

yield a two-parameter group of the third type, because the basis transformation symbols, viz.,

$$V_1f \equiv U_2f = x\frac{\partial f}{\partial y}, \qquad (3\text{-}106)$$

and

$$V_2f \equiv -U_1f = -x\frac{\partial f}{\partial x} \qquad (3\text{-}107)$$

satisfy Eqs. (3-96) and (3-97). The symbols of the once-extended groups of point transformations from Eqs. (3-104) and (3-105) are

$$U_1'f = x\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'}, \qquad (3\text{-}108)$$

and

$$U_2'f = x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}. \qquad (3\text{-}109)$$

The second-order differential equation

$$x^2y'' = F(xy' - y), \qquad (3\text{-}110)$$

and corresponding partial differential equation

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{F(xy' - y)}{x^2}\frac{\partial f}{\partial y'} = 0 \qquad (3\text{-}111)$$

are invariant under the group represented by the symbols of Eqs. (3-104) and (3-105), because the commutators formed with Eqs. (3-108), (3-109), and (3-111) are

$$(U_1'A)f = -Af, \qquad (3\text{-}112)$$

and

$$(U_2'A)f = 0. \qquad (3\text{-}113)$$

3.4 Differential Equations That Are Admitted by Two-Parameter Lie Groups of the Fourth Type

In this section we shall consider four two-parameter Lie groups of the fourth type. The symbols of the two infinitesimal transformations of these groups are

(a) for the first group,

$$U_1f = x\frac{\partial f}{\partial x}, \qquad (3\text{-}114)$$

and

$$U_2f = \frac{\partial f}{\partial x}; \qquad (3\text{-}115)$$

(b) for the second group,

$$U_1f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}, \qquad (3\text{-}116)$$

and

$$U_2f = x\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}; \qquad (3\text{-}117)$$

(c) for the third group,

$$U_1f = \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \qquad (3\text{-}118)$$

and

$$U_2f = x\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y}; \qquad (3\text{-}119)$$

(d) for the fourth group,

$$U_1f = \frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}, \qquad (3\text{-}120)$$

and

$$U_2f = x\frac{\partial f}{\partial x} + x^2\frac{\partial f}{\partial y}. \qquad (3\text{-}121)$$

These four two-parameter Lie groups of point transformations are all of the fourth type, because each satisfies the relations

$$(U_1U_2)f = U_1f, \qquad (3\text{-}122)$$

and

$$U_2f = \rho(x,y)U_1f, \qquad (3\text{-}123)$$

which define the fourth type of two-parameter Lie group.

The four second-order, ordinary differential equations that follow are invariant under the above four groups.

(a) The group of Eqs. (3-114) and (3-115) admits

$$y'' = (y')^2F(y); \qquad (3\text{-}124)$$

(b) That of Eqs. (3-116) and (3-117) admits

$$y'' = (y' - 1)^2F(y - x); \qquad (3\text{-}125)$$

(c) That of Eqs. (3-118) and (3-119) admits

$$y'' = \frac{(y')^2}{y} + y\left(\frac{y'}{y} - 1\right)^2F(\ln y - x); \qquad (3\text{-}126)$$

(d) That of Eqs. (3-120) and (3-121) admits

$$y'' = 1 + 2(y' - x)^2F(2y - x^2). \qquad (3\text{-}127)$$

In Eqs. (3-124) through (3-127), the symbol, F, denotes an arbitrary function of the indicated argument.

To prove the invariance assertion just made relative to Eq. (3-124), we first note that its equivalent first-order, linear partial differential equation is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + (y')^2F(y)\frac{\partial f}{\partial y'} = 0. \qquad (3\text{-}128)$$

The once-extended groups arising from the symbols of Eqs. (3-114) and (3-115) are generated by the symbols

$$U_1'f = x\frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y'}, \qquad (3\text{-}129)$$

and

$$U_2'f = \frac{\partial f}{\partial x}. \qquad (3\text{-}130)$$

The commutators formed from Eqs. (3-128) through (3-130) have the values

$$(U_1'A)f = -Af, \qquad (3\text{-}131)$$

and

$$(U_2'A)f = 0, \qquad (3\text{-}132)$$

which completes the proof.

The proofs for the invariance assertions made above for Eqs. (3-125) through (3-127) follow the same pattern as that contained in Eqs. (3-128) through (3-132), and we shall merely summarize the relevant relations below.

(1) For Eqs. (3-116), (3-117), and (3-125) we have

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \left[(y' - 1)^2F(y - x)\right]\frac{\partial f}{\partial y'} = 0, \qquad (3\text{-}133)$$

$$U_1'f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}, \qquad (3\text{-}134)$$

$$U_2'f = x\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} + (1 - y')\frac{\partial f}{\partial y'}, \qquad (3\text{-}135)$$

$$(U_1'A)f = 0, \qquad (3\text{-}136)$$

and

$$(U_2'A)f = -Af. \qquad (3\text{-}137)$$

(2) For Eqs. (3-118), (3-119), and (3-126) we have

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \left[\frac{(y')^2}{y} + y\left(\frac{y'}{y} - 1\right)^2F(\ln y - x)\right]\frac{\partial f}{\partial y'}$$
$$= 0 \qquad (3\text{-}138)$$

$$U_1'f = \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'}, \qquad (3\text{-}139)$$

$$U_2'f = x\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y} + \left[y + (x - 1)y'\right]\frac{\partial f}{\partial y'}, \qquad (3\text{-}140)$$

$$(U_1'A)f = 0, \qquad (3\text{-}141)$$

and

$$(U_2'A)f = -Af. \qquad (3\text{-}142)$$

(3) For Eqs. (3-120), (3-121), and (3-127) we have

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \left[1 + 2(y' - x)^2F(2Y - x^2)\right]\frac{\partial f}{\partial y'}$$
$$= 0, \qquad (3\text{-}143)$$

$$U_1'f = \frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}, \qquad (3\text{-}144)$$

$$U_2'f = x\frac{\partial f}{\partial x} + x^2\frac{\partial f}{\partial y} + (2x - y')\frac{\partial f}{\partial y'}, \qquad (3\text{-}145)$$

$$(U_1'A)f = 0, \qquad (3\text{-}146)$$

and

$$(U_2'A)f = -Af. \qquad (3\text{-}147)$$

The principal results obtained in Sections 3.2 through 3.4 are summarized in Table III, which gives the symbols of the infinitesimal transformations of 16 two-parameter Lie groups, the group type, and the second-order, ordinary differential equations that are admitted by these groups. In Table III, the following definitions are employed:

$$p \equiv \frac{\partial f}{\partial x}, \qquad (3\text{-}148)$$

and

$$q \equiv \frac{\partial f}{\partial y}. \qquad (3\text{-}149)$$

4. APPLICATION OF LIE GROUPS TO THE SOLUTION OF NONLINEAR DIFFUSION EQUATIONS

In the ensuing part, explicit, exact solutions of nonlinear diffusion equations will be found by exploiting the fact that these equations are invariant under Lie groups. Particular emphasis is given to establishing relationships among the parameters appearing in the nonlinear diffusion equations that must be satisfied for the solutions to exist together with solution integrals.

Table III. SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS ADMITTED BY VARIOUS TWO-PARAMETER LIE GROUPS NOT IN CANONICAL FORM

Group Symbols	Group Type	Invariant Differential Equation
$U_1f = q;\ U_2f = xp$	1	$x^2y'' = F(xy')$
$U_1f = p;\ U_2f = yq$	1	$y'' = yF\left(\frac{y'}{y}\right)$
$U_1f = xq;\ U_2f = xp + yq$	1	$xy'' = F\left(y' - \frac{y}{x}\right)$
$U_1f = xp;\ U_2f = yq$	1	$x^2y'' = yF\left(\frac{xy'}{y}\right)$
$U_1f = xq;\ U_2f = x^2q$	2	$x^2y'' - 2xy' + 2y = F(x)$
$U_1f = xyq;\ U_2f = yq$	2	$yy'' = (y')^2 + F(x)y^2$
$U_1f = \frac{q}{y};\ U_2f = \frac{xq}{y}$	2	$yy'' + (y')^2 = F(x)$
$U_1f = xy^2q;\ U_2f = x^2y^2q$	2	$yy'' - 2(y')^2 - \frac{2}{x}yy' - \frac{2}{x^2}y^2 = F(x)y^3$
$U_1f = p;\ U_2f = xp + yq$	3	$yy'' = F(y')$
$U_1f = q;\ U_2f = xp + nyq$	3	$y'' = x^{n-2}F(x^{1-n}y')$
$U_1f = xp + nyq;\ U_2f = xq$	3	$y'' = x^{n-2}F\left(\frac{y'}{x^{n-1}} - \frac{y}{x^n}\right)$
$U_1f = xp;\ U_2f = xq$	3	$x^2y'' = F(xy' - y)$
$U_1f = xp;\ U_2f = p$	4	$y'' = (y')^2F(y)$
$U_1f = p + q;\ U_2f = xp + yq$	4	$y'' = (y' - 1)^2F(y - x)$
$U_1f = p + yq;\ U_2f = xp + xyq$	4	$y'' = \frac{(y')^2}{y} + y\left(\frac{y'}{y} - 1\right)^2F(\ln y - x)$
$U_1f = p + xq;\ U_2f = xp + x^2q$	4	$y'' = 1 + 2(y' - x)^2F(2y - x^2)$

4.1 Group Invariance Properties of $y^\lambda y'' + \lambda y^{\lambda-1}(y')^2 + \alpha y^\nu = 0$

The nonlinear diffusion equation

$$y^\lambda y'' + \lambda y^{\lambda-1}(y')^2 + \alpha y^\nu = 0 \tag{4-1}$$

arises when the transport coefficient is proportional to the λ-power of the dependent variable, and the source term is proportional to the ν-power of the dependent variable. This nonlinear differential equation is invariant under the two-parameter Lie group, which is represented by the two infinitesimal transformation symbols

$$U_1f = \frac{\partial f}{\partial x}, \tag{4-2}$$

and

$$U_2f = x\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y}, \tag{4-3}$$

where in Eq. (4-3) the definition that follows has been introduced:

$$n \equiv \frac{2}{1 + \lambda - \nu}. \tag{4-4}$$

Accordingly, Eq. (4-1) is admitted by the group defined by Eqs. (4-2) and (4-3) provided that $\nu \neq 1 + \lambda$. This exceptional case will be discussed in Sec. 4.3.

The proof of the invariance assertion just made relative to Eq. (4-1) may be accomplished in two stages. First, it must be shown that the group represented by the symbols of Eqs. (4-2) and (4-3) is, in fact, a two-parameter Lie group. Second, the invariance of Eq. (4-1) under this group may be established by demonstrating the invariance of its equivalent first-order, linear partial differential equation under the group.

The fact that the symbols of Eqs. (4-2) and

(4-3) do generate a two-parameter Lie group is a direct consequence of Lie's principal theorem (cf. Sec. 2.1) because their commutator assumes the value

$$(U_1U_2)f = U_1f. \qquad (4\text{-}5)$$

Moreover, because these two symbols also satisfy the relation

$$U_2f \neq \rho(x,y)U_1f, \qquad (4\text{-}6)$$

the two-parameter group of point transformations generated by the two basis transformations of Eqs. (4-2) and (4-3) is, in fact, a two-parameter Lie group of the third type. The identification of the group type reduces the problem of the integration of Eq. (4-1) to one of quadratures.

The linear, first-order partial differential equation that corresponds to Eq. (4-1) is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} - \left\{\lambda y^{\lambda-1}(y')^2 + \alpha y^{\nu-\lambda}\right\}\frac{\partial f}{\partial y'} = 0, \qquad (4\text{-}7)$$

and the symbols of the once-extended groups of point transformations that come out of Eqs. (4-2) and (4-3) are

$$U_1'f = \frac{\partial f}{\partial x}, \qquad (4\text{-}8)$$

and

$$U_2'f = x\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y} + (n-1)y'\,\frac{\partial f}{\partial y'}, \qquad (4\text{-}9)$$

respectively. The commutator constructed with the differential operators appearing in Eqs. (4-7) and (4-8) is

$$(U_1'A)f = 0. \qquad (4\text{-}10)$$

The commutator from Eqs. (4-7) and (4-9) is

$$(U_2'A)f = \left[U_2'(1) - A(x)\right]\frac{\partial f}{\partial x} + \left[U_2'(y') - A(ny)\right]\frac{\partial f}{\partial y}$$
$$- \left\{U_2'\left[\lambda y^{-1}(y')^2 + \alpha y^{\nu-\lambda}\right]\right.$$
$$\left. + (n-1)A(y')\right\}\frac{\partial f}{\partial y'}. \qquad (4\text{-}11)$$

Now, because

$$U_2'(1) - A(x) = -1, \qquad (4\text{-}12)$$

$$U_2'(y') - A(ny) = -y', \qquad (4\text{-}13)$$

and

$$U_2'\left[\lambda y^{-1}(y')^2 + \alpha y^{\nu-\lambda}\right] + (n-1)A(y')$$
$$= ny\left[-\lambda y^{-2}(y')^2 + \alpha(\nu-\lambda)y^{\nu-\lambda-1}\right]$$
$$+ (n-1)\lambda 2y^{-1}(y')^2$$
$$- (n-1)\left[\lambda y^{-1}(y')^2 + \alpha y^{\nu-\lambda}\right], \qquad (4\text{-}14a)$$
$$= \lambda y^{-1}(y')^2 - \left[n(\nu-\lambda) - n + 1\right]\alpha y^{\nu-\lambda}, \qquad (4\text{-}14b)$$
$$= \lambda y^{-1}(y')^2 + \alpha y^{\nu-\lambda}, \qquad (4\text{-}14c)$$

where Eq. (4-14c) follows upon substituting Eq. (4-4) into Eq. (4-14b), we find that Eq. (4-11) reduces to

$$(U_2'A)f = -Af. \qquad (4\text{-}15)$$

From Eqs. (4-10) and (4-15) it follows that the linear partial differential equation in Eq. (4-7) is invariant under the once-extended groups represented by the symbols of Eqs. (4-8) and (4-9). Therefore, the nonlinear diffusion equation of Eq. (4-1) is invariant under the two-parameter Lie group of the third type whose basis transformation symbols are contained in Eqs. (4-2) and (4-3) provided that $\nu \neq 1 + \lambda$.

In the case when $\nu = 1 + \lambda$, Eq. (4-1) simplifies to the form

$$yy'' + \lambda(y')^2 + \alpha y^2 = 0, \qquad (4\text{-}16)$$

which is invariant under the two-parameter Lie group whose basis transformations are represented by the symbols

$$U_1f = \frac{\partial f}{\partial x}, \qquad (4\text{-}17)$$

and

$$U_2f = y\frac{\partial f}{\partial y}. \qquad (4\text{-}18)$$

This two-parameter group is of the first type because the symbols of Eqs. (4-17) and (4-18) satisfy the defining relations

$$(U_1U_2)f = 0, \qquad (4\text{-}19)$$

and

$$U_2f \neq \rho(x,y)U_1f. \qquad (4\text{-}20)$$

The symbols of the once-extended groups arising from Eqs. (4-17) and (4-18) are

$$U_1'f = \frac{\partial f}{\partial x}, \qquad (4\text{-}21)$$

and

$$U_2'f = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'}, \qquad (4\text{-}22)$$

and the linear partial differential equation equivalent to Eq. (4-16) is

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} - \left[\lambda y^{-1}(y')^2 + \alpha y\right]\frac{\partial f}{\partial y'} = 0. \qquad (4\text{-}23)$$

The commutators formed from the differential operators in Eqs. (4-21) through (4-23) reduce to

$$(U_1'A)f = 0, \qquad (4\text{-}24)$$

and

$$(U_2'A)f = 0, \qquad (4\text{-}25)$$

which establishes the fact that the two-parameter Lie group of the first type with the infinitesimal transformations of Eqs. (4-17) and (4-18) admits the nonlinear second-order differential equation of Eq. (4-16). Accordingly, the integration of Eq. (4-16) may be reduced to a quadrature problem, which is worked out in Sec. 4.3.

4.2 Introduction of Canonical Variables

It has been shown that Eqs. (4-2) and (4-3) comprise the basis transformations of a two-parameter Lie group of the third type that admits Eq. (4-1) if $\nu \neq 1 + \lambda$. Through the introduction of the canonical variables, which we shall denote by X and Y, these two infinitesimal transformations may be brought into their canonical forms, viz.,

$$\overline{U}_1 f = \frac{\partial f}{\partial Y}, \tag{4-26}$$

and

$$\overline{U}_2 f = X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} \tag{4-27}$$

for a two-parameter Lie group of the third type. Furthermore, if the nonlinear diffusion equation given in Eq. (4-1) is expressed in terms of canonical variables, then the resulting differential equation will be invariant under the two-parameter group in canonical variables generated by Eqs. (4-26) and (4-27). It is known from Table II that Eq. (4-1) will assume the general form

$$X\frac{d^2Y}{dX^2} = F\left(\frac{dY}{dX}\right), \tag{4-28}$$

where F signifies an arbitrary function of the argument when it is written in terms of the canonical variables of the two-parameter Lie group under which it is invariant. A first integral of Eq. (4-28) may be easily obtained by quadrature.

If

$$U_1 f = \xi_1(x,y)\frac{\partial f}{\partial x} + \eta_1(x,y)\frac{\partial f}{\partial y}, \tag{4-29}$$

and

$$U_2 f = \xi_2(x,y)\frac{\partial f}{\partial x} + \eta_2(x,y)\frac{\partial f}{\partial y} \tag{4-30}$$

are the symbols of the infinitesimal transformations of a two-parameter Lie group of the third type not in canonical form, then the canonical variables of this group may be determined[3] from

$$d \ln X = \frac{-\eta_1 dx + \xi_1 dy}{\eta_2\xi_1 - \eta_1\xi_2}, \tag{4-31}$$

which will be an exact differential, and from

$$\frac{\partial Y}{\partial x} = \frac{\eta_2 - \eta_1 Y}{\xi_1\eta_2 - \xi_2\eta_1}, \tag{4-32}$$

and

$$\frac{\partial Y}{\partial y} = \frac{-\xi_2 + \xi_1 Y}{\xi_1\eta_2 - \xi_2\eta_1}. \tag{4-33}$$

Relative to these last three relations, Eq. (4-31) is obtained upon solving

$$\xi_1(x,y)\frac{\partial X}{\partial x} + \eta_1(x,y)\frac{\partial X}{\partial y} = 0, \tag{4-34}$$

and

$$\xi_2(x,y)\frac{\partial X}{\partial x} + \eta_2(x,y)\frac{\partial X}{\partial y} = X, \tag{4-35}$$

for X_x and X_y, whereas Eqs. (4-32) and (4-33) are the solutions of

$$\xi_1(x,y)\frac{\partial Y}{\partial x} + \eta_1(x,y)\frac{\partial Y}{\partial y} = 1, \tag{4-36}$$

and

$$\xi_2(x,y)\frac{\partial Y}{\partial x} + \eta_2(x,y)\frac{\partial Y}{\partial y} = Y. \tag{4-37}$$

For the two-parameter group generated by the symbols of Eqs. (4-2) and (4-3), we find that Eqs. (4-31) through (4-33) become

$$d \ln X = \frac{dy}{ny}, \tag{4-38}$$

$$\frac{\partial Y}{\partial x} = 1, \tag{4-39}$$

and

$$\frac{\partial Y}{\partial y} = \frac{Y - x}{ny}, \tag{4-40}$$

respectively. Integrating Eq. (4-38) produces

$$X = y^{1/n}, \tag{4-41}$$

whereas

$$Y = x \tag{4-42}$$

obviously satisfies Eqs. (4-39) and (4-40). Accordingly, the results given in Eqs. (4-41) and (4-42) may be taken as the canonical variables of the two-parameter Lie group of the third type generated by the infinitesimal transformations contained in Eqs. (4-2) and (4-3).

In transforming Eq. (4-1) to canonical variables, we shall regard the canonical variable, Y, as the new dependent variable and the canonical variable, X, as the new independent variable. From Eq. (4-42), it will be noted that the dependent and independent variables are interchanged in the course of the reduction of Eq. (4-1) to its canonical form.

To effect this reduction we may first observe that

$$\frac{dY}{dX} = \frac{\frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial y}y'}{\frac{\partial X}{\partial x} + \frac{\partial X}{\partial y}y'} \tag{4-43}$$

becomes

$$\frac{dY}{dX} = \frac{n}{y'}\, y^{1-(1/n)} \tag{4-44}$$

when the partial derivatives are evaluated from Eqs. (4-41) and (4-42). Next we note that

$$\frac{d}{dx}\frac{dY}{dx} = \frac{dX}{dx}\frac{d^2Y}{dX^2} = n\,\frac{d}{dx}\left[(y')^{-1}\, y^{1-(1/n)}\right]. \tag{4-45}$$

However,

$$\frac{dX}{dx} = \frac{y'}{n}\, y^{(1/n)-1}, \tag{4-46}$$

so that Eq. (4-45) becomes

$$\frac{y'}{n}\, y^{(1/n)-1}\, \frac{d^2Y}{dX^2}$$

$$= (n-1)y^{-(1/n)} - ny^{1-(1/n)}(y')^{-2}y''. \tag{4-47}$$

Now, because

$$x = Y, \tag{4-48}$$

and

$$y = X^n, \tag{4-49}$$

we find that Eqs. (4-44) and (4-47) yield

$$y' = nX^{n-1}\left(\frac{dY}{dX}\right)^{-1}, \tag{4-50}$$

and

$$y'' = n(n-1)X^{n-2}\left(\frac{dY}{dX}\right)^{-2} - nX^{n-1}\left(\frac{dY}{dX}\right)^{-3}\frac{d^2Y}{dX^2} \tag{4-51}$$

respectively. Substituting Eqs. (4-49) through (4-51) into Eq. (4-1) produces, first of all,

$$n(n-1)X^{n\lambda+n-2}\left(\frac{dY}{dX}\right)^{-2} - nX^{n\lambda+n-1}\left(\frac{dY}{dX}\right)^{-3}\frac{d^2Y}{dX^2}$$

$$+ \lambda n^2 X^{n(\lambda-1)+2n-2}\left(\frac{dY}{dX}\right)^{-2} + \alpha X^{n\nu} = 0, \tag{4-52}$$

and simplifying this relation with the help of Eq. (4-4) yields

$$X\,\frac{d^2Y}{dX^2} = \left[n(1+\lambda) - 1\right]\frac{dY}{dX} + \frac{\alpha}{n}\left(\frac{dY}{dX}\right)^3. \tag{4-53}$$

This last equation is Eq. (4-1) written in terms of the canonical variables of the two-parameter Lie group with the basis transformation symbols given in Eqs. (4-2) and (4-3), and it is seen to have the general form anticipated in Eq. (4-28).

The result obtained in Eq. (4-53) is valid provided that $\nu \neq \lambda + 1$. In the case $\nu = \lambda + 1$, Eq. (4-1) simplifies to Eq. (4-16), which has been shown to be invariant under the two-parameter group of the first, not third, type generated by the infinitesimal transformations of Eqs. (4-17) and (4-18). If Eqs. (4-29) and (4-30) are now regarded as the symbols of the basis transformations of a two-parameter Lie group of the first type, then the canonical variables of this group may be found from the relations

$$dX = \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy, \tag{4-54}$$

and

$$dY = \frac{\partial Y}{\partial x}dx + \frac{\partial Y}{\partial y}dy. \tag{4-55}$$

In Eq. (4-54), the partial derivatives are solutions of the set

$$\xi_1(x,y)\frac{\partial X}{\partial x} + \eta_1(x,y)\frac{\partial X}{\partial y} = 1, \tag{4-56}$$

and

$$\xi_2(x,y)\frac{\partial X}{\partial x} + \eta_2(x,y)\frac{\partial X}{\partial y} = 0, \tag{4-57}$$

and in Eq. (4-55), of the set

$$\xi_1(x,y)\frac{\partial Y}{\partial x} + \eta_1(x,y)\frac{\partial Y}{\partial y} = 0, \tag{4-58}$$

and

$$\xi_2(x,y)\frac{\partial Y}{\partial x} + \eta_2(x,y)\frac{\partial Y}{\partial y} = 1. \tag{4-59}$$

For the basis transformations of Eqs. (4-17) and (4-18), it is found that Eqs. (4-56) through (4-59) become

$$\frac{\partial X}{\partial x} = 1, \tag{4-60}$$

$$y\frac{\partial X}{\partial y} = 0, \tag{4-61}$$

$$\frac{\partial Y}{\partial x} = 0, \tag{4-62}$$

and

$$y\frac{\partial Y}{\partial y} = 1, \tag{4-63}$$

respectively. Accordingly, Eqs. (4-54) and (4-55) assume the forms

$$dX = dx, \tag{4-64}$$

and

$$dY = \frac{dy}{y}, \tag{4-65}$$

which integrate out to produce

$$X = x, \tag{4-66}$$

and

$$Y = \ln y \tag{4-67}$$

as the canonical variables of the two-parameter Lie group generated by the basis transformation symbols of Eqs. (4-17) and (4-18).

If the nonlinear differential equation contained in Eq. (4-16) is written in terms of the canonical variables obtained in Eqs. (4-66) and (4-67), it will assume, in accordance with Table II, the general form

$$\frac{d^2Y}{dX^2} = F\left(\frac{dY}{dX}\right), \tag{4-68}$$

where F is an arbitrary function of the indicated argument, which is invariant under the canonical form of the two-parameter group arising from Eqs. (4-17) and (4-18), viz.,

$$\overline{U}_1 f = \frac{\partial f}{\partial X}, \qquad (4\text{-}69)$$

and

$$\overline{U}_2 f = \frac{\partial f}{\partial Y}. \qquad (4\text{-}70)$$

To reduce Eq. (4-16) to its canonical form, we first note that Eqs. (4-66) and (4-67) imply that

$$y = \exp\,(Y), \qquad (4\text{-}71)$$

$$y' = \exp\,(Y)\,\frac{dY}{dX}, \qquad (4\text{-}72)$$

and

$$y'' = \exp\,(Y)\left[\frac{d^2Y}{dX^2} + \left(\frac{dY}{dX}\right)^2\right]. \qquad (4\text{-}73)$$

Substituting Eqs. (4-71) through (4-73) into Eq. (4-16) and simplifying the result produces

$$\frac{d^2Y}{dX^2} + (1+\lambda)\left(\frac{dY}{dX}\right)^2 + \alpha = 0, \qquad (4\text{-}74)$$

which is the canonical form of Eq. (4-16), as anticipated in Eq. (4-68), and also the canonical form of Eq. (4-1) when $\nu = 1 + \lambda$.

We turn now to the integration of the two canonical forms that have been obtained above for Eq. (4-1), that is, to Eq. (4-74) when $\nu = 1 + \lambda$ and to Eq. (4-53) when $\nu \neq 1 + \lambda$.

4.3 Solution for the Case when $\nu = 1 + \lambda$

If we let

$$u \equiv \frac{dY}{dX} \qquad (4\text{-}75)$$

in Eq. (4-74), then the first integral is immediately obtained upon integration in the form,

$$\frac{dY}{dX} = A \tan\left[C_1 - A(1+\lambda)X\right], \qquad (4\text{-}76)$$

wherein C_1 is an arbitrary constant, and

$$A \equiv \sqrt{\frac{\alpha}{1 + \lambda}}. \qquad (4\text{-}77)$$

The integral of Eq. (4-76) may be written as

$$Y = \frac{1}{1+\lambda} \ln \cos\left[C_1 - A(1+\lambda)X\right] + \frac{1}{1+\lambda} \ln C_2 \qquad (4\text{-}78)$$

with C_2 as a second arbitrary constant. Upon reverting back to the original variables by means of Eqs. (4-66) and (4-67), Eq. (4-78) becomes

$$(1+\lambda) \ln y = \ln \cos\,[C_1 - A(1+\lambda)x] + \ln C_2, \qquad (4\text{-}79)$$

that is

$$y^{1+\lambda} = C_2 \cos\,[C_1 - A(1+\lambda)x]. \qquad (4\text{-}80)$$

This last equation is the general solution of Eq. (4-1) when $\nu = 1 + \lambda$, and, therefore, of Eq. (4-16). The solution given in Eq. (4-80) does not hold, however, if $\lambda = -1$. Solutions of Eq. (4-1) valid for $\lambda = -1$ and $\nu \neq 0$ and for $\lambda = -1$ and $\nu = 0$ will be given later (see Secs. 4.5.6 and 4.5.7).

The two arbitrary constants in Eq. (4-80) may be evaluated once the boundary or initial conditions are specified. For example, for the nonlinear, two-point boundary value problem such that

$$y'(0) = 0, \qquad (4\text{-}81)$$

and

$$y(1) = 1, \qquad (4\text{-}82)$$

we find that

$$C_1 = 0, \qquad (4\text{-}83)$$

and

$$C_2 = \frac{1}{\cos\,[A(1+\lambda)]}. \qquad (4\text{-}84)$$

Accordingly, it follows that

$$y = \left(\frac{\cos\,[x\sqrt{\alpha(1+\lambda)}]}{\cos\,[\sqrt{\alpha(1+\lambda)}]}\right)^{\frac{1}{1+\lambda}} \qquad (4\text{-}85)$$

is the solution of Eq. (4-1) when $\nu = 1 + \lambda$ provided that $\lambda \neq -1$, $y'(0) = 0$, and $y(1) = 1$, i.e., of Eq. (4-16) under these same conditions.

4.4 Quadrature Formula for the Case when $\nu \neq 1 + \lambda$

In this section an integral representation will be derived for the solution of Eq. (4-1) when $\nu \neq 1 + \lambda$ and for the nonlinear two-point boundary value problem with the boundary conditions as given in Eqs. (4-81) and (4-82). We may state that an infinite number of solutions of Eq. (4-1) may be found with the integral representation to be derived. However, we shall limit ourselves to the subsequent consideration of approximately two dozen cases that effectively provide the solution of Eq. (4-1) for a continuous variation of the parameters, λ and ν, that appear in this equation.

To find the integral representation mentioned above, we begin with the canonical form of Eq. (4-1), that is, with Eq. (4-53), which, with the definition contained in Eq. (4-75), may be written as

$$\frac{du}{u(u^2 + a^2)} = \frac{\alpha}{n}\,\frac{dX}{X}, \qquad (4\text{-}86)$$

wherein

$$a^2 \equiv \frac{n}{\alpha}\,[n(1+\lambda) - 1]. \qquad (4\text{-}87)$$

To express the boundary conditions, $y'(0) = 0$ and

$y(1) = 1$, in terms of the canonical variables, let $y(0) = y_o$ be the value of the solution of Eq. (4-1) at $x = 0$. Then from Eqs. (4-48) and (4-49) it follows that

$$Y = 0 \tag{4-88}$$

and

$$X = y_o^{1/n} \tag{4-89}$$

and from Eq. (4-44)

$$\frac{dY}{dX} \xrightarrow[x \to 0]{} \infty. \tag{4-90}$$

Hence, in integrating Eq. (4-86), use is made of the fact that $u \to \infty$ as $X \to y_o^{1/n}$ for the two-point boundary conditions of Eqs. (4-81) and (4-82), that is,

$$\int_u^{\infty} \frac{1}{u'[(u')^2 + a^2]} du' = \frac{\alpha}{n} \int_X^{y_o^{1/n}} \frac{1}{X'} dX'. \tag{4-91}$$

This integrates out to the form

$$\ln\left(\frac{u^2}{u^2 + a^2}\right) = \frac{2a^2\alpha}{n} \ln\left(\frac{X}{y_o^{1/n}}\right), \tag{4-92}$$

or

$$\frac{u^2}{u^2 + a^2} = \left(\frac{X}{y_o^{1/n}}\right)^m \tag{4-93}$$

with the definition

$$m \equiv \frac{2\alpha a^2}{n} = 2[n(1+\lambda) - 1]. \tag{4-94}$$

Now from Eq. (4-93) we obtain

$$\frac{dY}{dX}^2 = \frac{a^2\left(\frac{X}{y_o^{1/n}}\right)^m}{1 - \left(\frac{X}{y_o^{1/n}}\right)^m}. \tag{4-95}$$

When taking the square root of Eq. (4-95) either the plus or the minus sign may be used. To decide which one to use, it may be noted that for the boundary conditions of Eqs. (4-81) and (4-82) the solution of Eq. (4-1) will be such that $y' \leq 0$ on the closed interval $0 \leq x \leq 1$. Accordingly, it may be concluded from Eq. (4-44) that the negative sign is to be used if n is positive, whereas the plus sign is to be used if n is negative. That is, Eq. (4-95) yields

$$\frac{dY}{dX} = \pm a \sqrt{\frac{\left(\frac{X}{y_o^{1/n}}\right)^m}{1 - \left(\frac{X}{y_o^{1/n}}\right)^m}} \tag{4-96}$$

in which we take the plus sign, if $n < 0$, and the minus sign, if $n > 0$. In Eq. (4-96), the boundary condition $y'(0) = 0$ has already been incorporated, and the arbitrary constant, y_o, is interpreted as the value of the solution of Eq. (4-1) at $x = 0$.

To put in the boundary condition $y(1) = 1$, we observe from Eqs. (4-48) and (4-49) that this boundary condition implies that $Y = 1$ when $X = 1$. Hence, Eq. (4-96) yields

$$\int_1^Y dY' = \pm a \int_1^X \sqrt{\frac{\left(\frac{X'}{y_o^{1/n}}\right)^m}{1 - \left(\frac{X'}{y_o^{1/n}}\right)^m}} dX', \tag{4-97}$$

or

$$\frac{1 - Y}{a} = \mp \int_1^X \sqrt{\frac{\left(\frac{X'}{y_o^{1/n}}\right)^m}{1 - \left(\frac{X'}{y_o^{1/n}}\right)^m}} dX', \tag{4-98}$$

in which the upper minus sign is now taken if $n < 0$, and the lower plus sign if $n > 0$. Upon letting

$$t = \frac{X'}{y_o^{1/n}}, \tag{4-99}$$

and upon reverting back to the original variables with Eqs. (4-48) and (4-49), we find that Eq. (4-98) becomes

$$\frac{1 - x}{a y_o^{1/n}} = \mp \int_{\left(\frac{1}{y_o}\right)^{1/n}}^{\left(\frac{y}{y_o}\right)^{1/n}} \sqrt{\frac{t^m}{1 - t^m}} dt \tag{4-100}$$

where the minus sign is used if $n < 0$, and the plus sign is used if $n > 0$.

In summary, Eq. (4-100) comprises an integral representation of the solution of Eq. (4-1) for the boundary conditions $y'(0) = 0$ and $y(1) = 1$. These boundary conditions have been incorporated into this equation, which is valid if $\nu \neq 1 + \lambda$.

The constant, y_o, in Eq. (4-100) is the value of the solution of Eq. (4-1) at $x = 0$ for the given boundary conditions and is determined as a root of the transcendental equation,

$$1 = \mp a\, y_o^{1/n} \int_{\left(\frac{1}{y_o}\right)^{1/n}}^{1} \sqrt{\frac{t^m}{1 - t^m}}\, dt, \qquad (4\text{-}101)$$

when this root exists. As will be seen later, Eq. (4-101) may have zero, one, or two roots. When this equation has no roots, as may happen for certain combinations of the values of the parameters λ, ν, and α appearing in Eq. (4-1), then this nonlinear diffusion equation has no solution for the boundary conditions $y'(0) = 0$ and $y(1) = 1$. Accordingly, we shall have to determine allowed values of the parameters for which a solution of Eq. (4-1) does, in fact, exist for the above stated boundary conditions.

4.5 Reductions of the Quadrature Formula in the Case when $\nu \neq 1 + \lambda$

In this section 25 explicit, analytic solutions of Eq. (4-1) will be found by specializing the integral representation obtained in Eq. (4-100), in which, from Eqs. (4-4), (4-87), and (4-94),

$$\frac{1}{n} = \frac{1 + \lambda - \nu}{2}, \qquad (4\text{-}102)$$

$$\frac{1}{a} = (1 + \lambda - \nu)\sqrt{\frac{\alpha}{2(1 + \lambda + \nu)}}, \qquad (4\text{-}103)$$

and

$$m = 2\left(\frac{1 + \lambda + \nu}{1 + \lambda - \nu}\right). \qquad (4\text{-}104)$$

4.5.1 Result for $m = 1$

If $m = 1$, then Eq. (4-104) implies that

$$\nu = \frac{-1}{3}(1 + \lambda), \qquad (4\text{-}105)$$

so $\nu > 0$, if $\lambda < -1$. Also,

$$n = \frac{3}{2(1 + \lambda)}, \qquad (4\text{-}106)$$

and

$$\frac{1}{a^2} = \frac{3}{4\alpha(1 + \lambda)}, \qquad (4\text{-}107)$$

so that, when $\lambda < -1$, we may write

$$\frac{1}{n} = \frac{-2(|\lambda| - 1)}{3}, \qquad (4\text{-}108)$$

and $a = iA$, where

$$\frac{1}{A^2} = \frac{4\alpha}{3}(|\lambda| - 1). \qquad (4\text{-}109)$$

Because $n < 0$, if $\lambda < -1$, for this case Eq. (4-100) becomes

$$\frac{1 - x}{iAy_o^{1/n}} = -\int_{\left(\frac{1}{y_o}\right)^{1/n}}^{\left(\frac{y}{y_o}\right)^{1/n}} \sqrt{\frac{t}{1 - t}}\, dt, \qquad (4\text{-}110a)$$

$$= -\int_{\left(\frac{1}{y_o}\right)^{1/n}}^{1} \sqrt{\frac{t}{1 - t}}\, dt - \int_{1}^{\left(\frac{y}{y_o}\right)^{1/n}} \sqrt{\frac{t}{1 - t}}\, dt. \qquad (4\text{-}110b)$$

The value of the solution at $x = 0$ is a root of

$$\frac{y_o^{-1/n}}{iA} = \int_{1}^{y_o^{-1/n}} \sqrt{\frac{t}{1 - t}}\, dt, \qquad (4\text{-}111)$$

and the solution itself is given by

$$\frac{xy_o^{-1/n}}{iA} = \int_{1}^{\left(\frac{y_o}{y}\right)^{-1/n}} \sqrt{\frac{t}{1 - t}}\, dt. \qquad (4\text{-}112)$$

Now, because

$$y_o^{-1/n} > 1, \qquad (4\text{-}113)$$

and

$$\left(\frac{y_o}{y}\right)^{-1/n} > 1, \qquad (4\text{-}114)$$

the denominators of the integrands in Eqs. (4-111) and (4-112) are written as

$$\sqrt{1 - t} = i\sqrt{t - 1}, \qquad (4\text{-}115)$$

and these equations become

$$\frac{y_o^{-1/n}}{A} = \int_{1}^{y_o^{-1/n}} \sqrt{\frac{t}{t - 1}}\, dt, \qquad (4\text{-}116)$$

and

$$\frac{xy_o^{-1/n}}{A} = \int_1^{\left(\frac{y_o}{y}\right)^{-1/n}} \sqrt{\frac{t}{t-1}}\, dt, \qquad (4\text{-}117)$$

respectively. Upon setting $t = \tau^2$ in Eqs. (4-116) and (4-117) and integrating out, we obtain

$$\frac{1}{A} = y_o^{1/2n}\sqrt{y_o^{-1/n} - 1} + y_o^{1/n}\,\mathrm{arccosh}\left(y_o^{-1/2n}\right), \qquad (4\text{-}118)$$

and

$$x = Ay_o^{1/n}\left\{\left(\frac{y_o}{y}\right)^{-1/2n}\sqrt{\left(\frac{y_o}{y}\right)^{-1/n} - 1} + \mathrm{arccosh}\left[\left(\frac{y_o}{y}\right)^{-1/2n}\right]\right\}. \qquad (4\text{-}119)$$

The right-hand side of Eq. (4-118) is plotted as a function of $y_o^{-1/2n}$ in Fig. 1. The occurrence of the maximum on this curve shows that

$$\left(\frac{1}{A}\right)_{max} = 1.20 \qquad (4\text{-}120)$$

defines a maximum value of α, denoted by α_{max} and found from Eqs. (4-109) and (4-120) to be

$$\alpha_{max} = \frac{1.08}{|\lambda| - 1}, \qquad (4\text{-}121)$$

for which there will be a root of Eq. (4-118). If $\alpha > \alpha_{max}$, then Eq. (4-1) has no solution when $\lambda < -1$ and

$$\nu = \frac{|\lambda| - 1}{3}. \qquad (4\text{-}122)$$

On the other hand, if $\alpha < \alpha_{max}$, and if

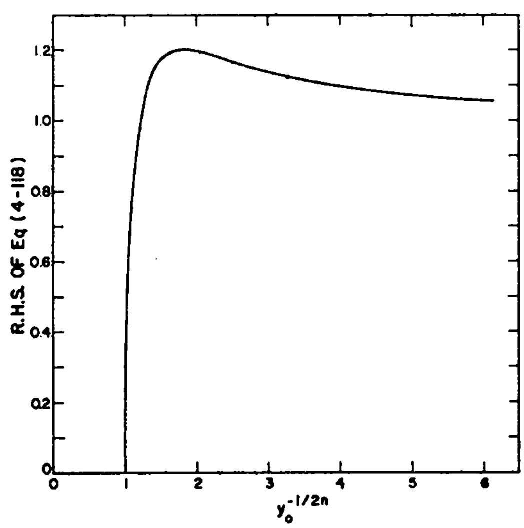

Fig. 1. Graph for determining the roots of Eq. (4-118).

$$1 < \frac{1}{A} < 1.20, \qquad (4\text{-}123)$$

then Eq. (4-118) has two roots, of which the smallest is of physical interest, and, if

$$\frac{1}{A} < 1 \qquad (4\text{-}124)$$

then Eq. (4-118) has a single root as is the case for $\alpha \mp \alpha_{max}$. When a root of Eq. (4-118) exists, Eq. (4-119) is the solution of Eq. (4-1) for $\lambda < -1$ subject to the condition of Eq. (4-122) and with the value of n given by Eq. (4-108) and that of A in Eq. (4-109).

<u>4.5.2 Results for m = 1/q, q = 1.5, 2, 2.5, 3, 3.5</u>

Let

$$m = \frac{1}{q}, \qquad (4\text{-}125)$$

then Eqs. (4-102) through (4-104) become

$$\frac{1}{n} = \frac{2q(1 + \lambda)}{1 + 2q}, \qquad (4\text{-}126)$$

$$\frac{1}{a^2} = \frac{4\alpha q^2(1 + \lambda)}{1 + 2q}, \qquad (4\text{-}127$$

and

$$\nu = \frac{(1 + \lambda)(1 - 2q)}{1 + 2q}, \qquad (4\text{-}128)$$

respectively. From Eq. (4-128) we see that, if $\lambda < -1$ and $q > 0.50$, then $\nu > 0$, and this is the case that will be considered. If $\lambda < -1$, Eq. (4-127) indicates that a is a pure imaginary number, so let $a = iA$ with

$$\frac{1}{A^2} = \frac{4\alpha q^2(|\lambda| - 1)}{1 + 2q}, \text{ for } \lambda < -1. \qquad (4\text{-}129)$$

Also, Eq. (4-126) shows that $n < 0$, if $\lambda < -1$. Because $n < 0$, Eq. (4-101) now takes the form

$$\frac{y_o^{-1/n}}{iA} = \int_1^{y_o^{-1/n}} \sqrt{\frac{t^{1/q}}{1 - t^{1/q}}}\, dt, \qquad (4\text{-}130)$$

or, because $y_o^{-1/n} > 1$,

$$\frac{y_o^{-1/n}}{A} = \int_1^{y_o^{-1/n}} \sqrt{\frac{t^{1/q}}{t^{1/q} - 1}}\, dt. \qquad (4\text{-}131)$$

Let $t = \tau^{2q}$ in Eq. (4-131), then we have

$$\frac{y_o^{-1/n}}{A} = 2q \int_1^{y_o^{-1/2qn}} \frac{\tau^{2q}}{\sqrt{\tau^2 - 1}} \, d\tau. \qquad (4\text{-}132)$$

By a procedure similar to that leading to Eq. (4-132), it is found that Eq. (4-100) reduces to

$$\frac{x}{A} y_o^{-1/n} = 2q \int_1^{\left(\frac{y_o}{y}\right)^{-1/2qn}} \frac{\tau^{2q}}{\sqrt{\tau^2 - 1}} \, d\tau. \qquad (4\text{-}133)$$

From Eq. (4-133) the following five solutions of Eq. (4-1) may be obtained by integration:

(1) if $q = 3/2$, then

$$x = A\, y_o^{1/n}\left\{\sqrt{\left(\frac{y_o}{y}\right)^{-2/3n} - 1}\ \left[\left(\frac{y_o}{y}\right)^{-2/3n} + 2\right]\right\}, \qquad (4\text{-}134)$$

in which y_o is a root of

$$\frac{1}{A} = \sqrt{1 - \frac{1}{y_o^{-2/3n}}}\ \left[1 + \frac{2}{y_o^{-2/3n}}\right], \qquad (4\text{-}135)$$

and

$$\frac{1}{n} = \frac{-3}{4}\,(|\lambda| - 1), \qquad (4\text{-}136)$$

and

$$\frac{1}{A^2} = \frac{9\alpha}{4}\,(|\lambda| - 1). \qquad (4\text{-}137)$$

This solution is also subject to the relation

$$\nu = \frac{|\lambda| - 1}{2}. \qquad (4\text{-}138)$$

(2) If $q = 2$, then

$$x = Ay_o^{1/n}\left\{\left(\frac{y_o}{y}\right)^{-1/4n}\sqrt{\left(\frac{y_o}{y}\right)^{-1/2n} - 1}\ \left[\left(\frac{y_o}{y}\right)^{-1/2n} + \frac{3}{2}\right]\right.$$
$$\left. + \frac{3}{2}\,\mathrm{arccosh}\left(\frac{y_o}{y}\right)^{-1/4n}\right\}, \qquad (4\text{-}139)$$

in which y_o is a root of

$$\frac{1}{A} = y_o^{1/n}\left\{y_o^{-1/4n}\sqrt{y_o^{-1/2n} - 1}\ \left[y_o^{-1/2n} + \frac{3}{2}\right]\right.$$
$$\left. + \frac{3}{2}\,\mathrm{arccosh}\ y_o^{-1/4n}\right\}, \qquad (4\text{-}140)$$

and

$$\frac{1}{n} = \frac{-4}{5}\,(|\lambda| - 1), \qquad (4\text{-}141)$$

$$\frac{1}{A^2} = \frac{16\alpha}{5}\,(|\lambda| - 1), \qquad (4\text{-}142)$$

and

$$\nu = \frac{3}{5}\,(|\lambda| - 1). \qquad (4\text{-}143)$$

(3) If $q = 5/2$, then

$$x = Ay_o^{1/n}\sqrt{\left(\frac{y_o}{y}\right)^{-2/5n} - 1}\left\{\left[\left(\frac{y_o}{y}\right)^{-2/5n} - 1\right]^2\right.$$
$$\left. + \frac{10}{3}\left[\left(\frac{y_o}{y}\right)^{-2/5n} - 1\right] + 5\right\}, \qquad (4\text{-}144)$$

in which y_o is a root of

$$\frac{1}{A} = y_o^{1/n}\sqrt{y_o^{-2/5n} - 1}\ \left[(y_o^{-2/5n} - 1)^2\right.$$
$$\left. + \frac{10}{3}\,(y_o^{-2/5n} - 1) + 5\right], \qquad (4\text{-}145)$$

and

$$\frac{1}{n} = \frac{-5}{6}\,(|\lambda| - 1), \qquad (4\text{-}146)$$

$$\frac{1}{A^2} = \frac{25\alpha}{6}\,(|\lambda| - 1), \qquad (4\text{-}147)$$

and

$$\nu = \frac{2}{3}\,(|\lambda| - 1). \qquad (4\text{-}148)$$

(4) If $q = 3$, then

$$x = Ay_o^{1/n}\left\{\sqrt{\left(\frac{y_o}{y}\right)^{-1/3n} - 1}\ \left[\left(\frac{y_o}{y}\right)^{-5/6n}\right.\right.$$
$$\left. + \frac{5}{4}\left(\frac{y_o}{y}\right)^{-1/2n} + \frac{15}{8}\left(\frac{y_o}{y}\right)^{-1/6n}\right]$$
$$\left. + \frac{15}{8}\,\mathrm{arccosh}\left(\frac{y_o}{y}\right)^{-1/6n}\right\}, \qquad (4\text{-}149)$$

in which y_o is a root of

$$\frac{1}{A} = y_o^{1/n}\left\{\sqrt{y_o^{-1/3n} - 1}\ \left[y_o^{-5/6n} + \frac{5}{4}\,y_o^{-1/2n}\right.\right.$$
$$\left.\left. + \frac{15}{8}\,y_o^{-1/6n}\right] + \frac{15}{8}\,\mathrm{arccosh}\ y_o^{-1/6n}\right\}, \qquad (4\text{-}150)$$

and

$$\frac{1}{n} = \frac{-6}{7}\,(|\lambda| - 1), \qquad (4\text{-}151)$$

$$\frac{1}{A^2} = \frac{36\alpha}{7}\,(|\lambda| - 1), \qquad (4\text{-}152)$$

and

$$\nu = \frac{5}{7}\,(|\lambda| - 1). \qquad (4\text{-}153)$$

(5) If $q = 7/2$, then

$$x = Ay_o^{1/n}\sqrt{\left(\frac{y_o}{y}\right)^{-2/7n} - 1}\left\{\left[\left(\frac{y_o}{y}\right)^{-2/7n} - 1\right]^3\right.$$
$$+ \frac{21}{5}\left[\left(\frac{y_o}{y}\right)^{-2/7n} - 1\right]^2$$
$$\left. + 7\left[\left(\frac{y_o}{y}\right)^{-2/7n} - 1\right] + 7\right\}, \qquad (4\text{-}154)$$

in which y_o is a solution of

$$\frac{1}{A} = y_o^{1/n} \sqrt{y_o^{-2/7n} - 1} \left[(y_o^{-2/7n} - 1)^3 + \frac{21}{5} (y_o^{-2/7n} - 1)^2 + 7 (y_o^{-2/7n} - 1) + 7 \right], \tag{4-155}$$

and

$$\frac{1}{n} = \frac{-7}{8} (|\lambda| - 1), \tag{4-156}$$

$$\frac{1}{A^2} = \frac{49\alpha}{8} (|\lambda| - 1), \tag{4-157}$$

and

$$\nu = \frac{3}{4} (|\lambda| - 1). \tag{4-158}$$

The five transcendental equations for determining the value, y_o, of the solution of Eq. (4-1) at $x = 0$ for the five solutions of this equation given above will have zero, one, or two roots. For example, the right-hand side of Eq. (4-135) is plotted in Fig. 2 as a function of $y_o^{-1/n}$. The maximum on the curve of this figure corresponds to

$$\left(\frac{1}{A}\right)_{max} = 1.42. \tag{4-159}$$

Hence, from Eqs. (4-137) and (4-159), it follows that

$$\alpha_{max} = \frac{0.896}{|\lambda| - 1} \tag{4-159a}$$

is the maximum allowed value of α for which a solution of Eq. (4-1) will exist when Eq. (4-138) holds. If $\alpha = \alpha_{max}$, there will be just one root found from Eq. (4-135). If

$$1 < \frac{1}{A} < 1.42, \tag{4-160}$$

then Eq. (4-135) has two roots, and if

$$\frac{1}{A} < 1, \tag{4-161}$$

one root. When two roots exist, the smaller is of physical interest, and when $\alpha \leq \alpha_{max}$, the corresponding solutions of Eq. (4-1) are those contained in Eq. (4-134).

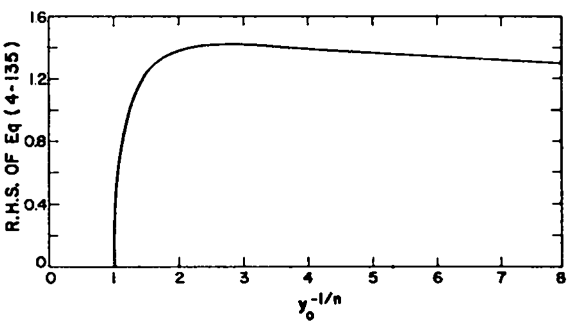

Fig. 2. Graph for determining the roots of Eq. (4-135).

Similar relations may be determined for the other four solutions embodied in Eqs. (4-139) through (4-158).

4.5.3 Results for m = 0

It is logical in the systematic development of solutions of Eq. (4-1) to discuss at this point the $m = 0$ case, even though the corresponding solutions do not come out of Eq. (4-100). If $m = 0$, then Eq. (4-104) indicates that

$$\nu = -(1 + \lambda), \tag{4-162}$$

and Eq. (4-1) assumes the form

$$y^{1+2\lambda} y'' + \lambda y^{2\lambda} (y')^2 + \alpha = 0. \tag{4-163}$$

We shall prove that the solution of Eq. (4-163) for the two-point boundary conditions, $y'(0) = 0$ and $y(1) = 1$, is

$$x = y_o^{1+\lambda} \sqrt{\frac{\pi}{2\alpha(1 + \lambda)}} \operatorname{erf} \sqrt{(1 + \lambda) \ln\left(\frac{y_o}{y}\right)}, \tag{4-164}$$

in which y_o is a root of

$$1 = y_o^{1+\lambda} \sqrt{\frac{\pi}{2\alpha(1 + \lambda)}} \operatorname{erf} \sqrt{(1 + \lambda) \ln y_o}, \tag{4-165}$$

provided that $\lambda > -1$. On the other hand, if $\lambda < -1$, then the solution of Eq. (4-163) for these same boundary conditions is

$$x = \frac{1}{y^{\ell-1}} \sqrt{\frac{2}{\alpha} \ln\left(\frac{y_o}{y}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j \left[2(\ell-1) \ln\left(\frac{y_o}{y}\right)\right]^j}{(1;2;j + 1)}, \tag{4-166}$$

in which $\ell \equiv |\lambda|$, and

$$(1;2;j + 1) \equiv 1 \cdot 3 \cdot 5 \cdot 7 \cdots (1 + 2j). \tag{4-167}$$

Also, in Eq. (4-166) the value, y_o, of the solution at $x = 0$ is a root of the transcendental equation

$$1 = \sqrt{\frac{2}{\alpha} \ln y_o} \sum_{j=0}^{\infty} \frac{(-1)^j [2(\ell-1) \ln y_o]^j}{(1;2;j + 1)}. \tag{4-168}$$

Neither the solution given in Eq. (4-164) nor that given in Eq. (4-166) is valid if $\lambda = -1$.

To prove these assertions it may be observed, first of all, that Eq. (4-163) is invariant under the two-parameter Lie group of point transformations generated by the infinitesimal transformations whose symbols are

$$U_1 f = \frac{\partial f}{\partial x}, \tag{4-169}$$

and

$$U_2f = x\frac{\partial f}{\partial x} + \frac{y}{1+\lambda}\frac{\partial f}{\partial y}. \qquad (4\text{-}170)$$

This invariance property is a direct consequence of the fact that the commutators evaluated from the differential operators appearing in the once-extended group symbols

$$U_1'f = \frac{\partial f}{\partial x}, \qquad (4\text{-}171)$$

and

$$U_2'f = x\frac{\partial f}{\partial x} + \frac{y}{1+\lambda}\frac{\partial f}{\partial y} - \frac{y'}{1+\lambda}\frac{\partial f}{\partial y'}, \qquad (4\text{-}172)$$

and the corresponding partial differential equation

$$Af \equiv \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} - \left[\lambda y^{-1}(y')^2 + \alpha y^{-1-2\lambda}\right]\frac{\partial f}{\partial y'} = 0, \qquad (4\text{-}173)$$

are found to be

$$(U_1'A)f = 0, \qquad (4\text{-}174)$$

and

$$(U_2'A)f = -Af. \qquad (4\text{-}175)$$

Second, the two-parameter group generated by the symbols of Eqs. (4-169) and (4-170) is of the third type with the canonical variables

$$X = y^{1+\lambda}, \qquad (4\text{-}176)$$

and

$$Y = x. \qquad (4\text{-}177)$$

To transform Eq. (4-163) to its canonical form, it may be noted that, in terms of the canonical variables,

$$y = X^{\frac{1}{1+\lambda}}, \qquad (4\text{-}178)$$

$$y' = \frac{1}{1+\lambda}X^{\frac{-\lambda}{1+\lambda}}\left(\frac{dY}{dX}\right)^{-1}, \qquad (4\text{-}179)$$

and

$$y'' = \frac{-\lambda}{(1+\lambda)^2}X^{\frac{-(1+2\lambda)}{1+\lambda}}\left(\frac{dY}{dX}\right)^{-2} - \frac{X^{\frac{-\lambda}{1+\lambda}}}{1+\lambda}\left(\frac{dY}{dX}\right)^{-3}\frac{d^2Y}{dX^2}. \qquad (4\text{-}180)$$

Upon introducing Eqs. (4-178) through (4-180) into Eq. (4-163), this latter equation assumes its canonical form, viz.,

$$X\frac{d^2Y}{dX^2} = \alpha(1+\lambda)\left(\frac{dY}{dX}\right)^3. \qquad (4\text{-}181)$$

A first integral of Eq. (4-181) is

$$\left(\frac{dY}{dX}\right)^2 = \frac{1}{C_1 - 2\alpha(1+\lambda)\ln X}, \qquad (4\text{-}182)$$

where C_1 is an arbitrary constant.

Now assume that $\lambda > -1$. Because $y' < 0$, Eq. (4-179) shows that the negative square root is to be taken in Eq. (4-182) if $\lambda > -1$, that is,

$$\frac{dY}{dX} = -\frac{1}{\sqrt{C_1 - 2\alpha(1+\lambda)\ln X}}, \qquad (4\text{-}183)$$

in this case. Upon reverting back to the original variables, Eq. (4-183) becomes

$$y' = \frac{-y^{-\lambda}}{1+\lambda}\sqrt{C_1 - 2\alpha(1+\lambda)^2\ln y}. \qquad (4\text{-}184)$$

Imposing the boundary condition, $y'(0) = 0$, yields

$$C_1 = 2\alpha(1+\lambda)^2\ln y_o, \qquad (4\text{-}185)$$

in which y_o is the value of the solution at $x = 0$, so that Eq. (4-184) reduces to

$$\frac{y^\lambda}{\sqrt{\ln(y_o/y)}}\,dy = -\sqrt{2\alpha}\,dx. \qquad (4\text{-}186)$$

Integrating Eq. (4-186) gives

$$\int_{y_o}^{y}\frac{t^\lambda}{\sqrt{\ln(y_o/t)}}\,dt = -\sqrt{2\alpha}\,x. \qquad (4\text{-}187)$$

To simplify Eq. (4-187), let

$$u^2 = \ln(y_o/t), \qquad (4\text{-}188)$$

then it becomes

$$x = y_o^{1+\lambda}\sqrt{\frac{2}{\alpha}}\int_0^{\sqrt{\ln(y_o/y)}}\exp\left[-(1+\lambda)u^2\right]du, \qquad (4\text{-}189)$$

which reduces to the solution given in Eq. (4-164) for $\lambda > -1$. The transcendental equation for y_o given in Eq. (4-165) is obtained from Eq. (4-189) by invoking the second boundary condition, $y(1) = 1$.

In the case when $\lambda < -1$, the positive square root is to be taken in Eq. (4-182), that is

$$\frac{dY}{dX} = \frac{1}{\sqrt{C_1 - 2\alpha(1+\lambda)\ln X}}. \qquad (4\text{-}190)$$

With $\ell = |\lambda|$, reverting Eq. (4-190) back to the original variables produces

$$dx = \frac{(1-\ell)y^{-\ell}}{\sqrt{C_1 - 2\alpha(\ell-1)^2\ln y}}\,dy, \qquad (4\text{-}191)$$

and imposing the boundary condition, $y'(0) = 0$, gives

$$C_1 = 2\alpha(\ell-1)^2\ln y_o. \qquad (4\text{-}192)$$

Consequently, Eq. (4-191) reduces to

$$\sqrt{2\alpha}\,dx = \frac{y^{-\ell}}{\sqrt{\ln(y_o/y)}}\,dy. \qquad (4\text{-}193)$$

With the boundary condition, $y(1) = 1$, Eq. (4-193) integrates out to

$$\sqrt{2\alpha}\,(1 - x) = \int_1^y \frac{t^{-\ell}}{\sqrt{\ln\,(y_o/t)}}\,dt. \qquad (4\text{-}194)$$

Because the value, y_o, of the solution at $x = 0$ may be calculated as a root of the transcendental equation

$$\sqrt{2\alpha} = \int_1^{y_o} \frac{t^{-\ell}}{\sqrt{\ln\,(y_o/t)}}\,dt, \qquad (4\text{-}195)$$

the solution of Eq. (4-163) for $\lambda < -1$ may be written in the form

$$\sqrt{2\alpha}\,x = \int_y^{y_o} \frac{t^{-\ell}}{\sqrt{\ln\,(y_o/t)}}\,dt. \qquad (4\text{-}196)$$

If we now set

$$\tau = (\ell-1)\ln\,(y_o/t) \qquad (4\text{-}197)$$

in Eqs. (4-196) and (4-195), we obtain

$$x = \frac{y_o^{1-\ell}}{\sqrt{2\alpha(\ell-1)}} \int_0^{(\ell-1)\ln\,(y_o/y)} \frac{\exp\,(\tau)}{\sqrt{\tau}}\,d\tau, \qquad (4\text{-}198)$$

and

$$\sqrt{2\alpha(\ell-1)} = y_o^{1-\ell} \int_0^{(\ell-1)\ln y_o} \frac{\exp\,(\tau)}{\sqrt{\tau}}\,d\tau. \qquad (4\text{-}199)$$

Evaluating the integral in Eq. (4-198) gives the series solution of Eq. (4-163) contained in Eq. (4-166) for $\lambda < -1$, and evaluating Eq. (4-199) produces Eq. (4-168), which completes the proofs for the solutions of Eq. (4-163).

4.5.4 Results for $m = -1/q$, $q = 1.5, 2, 2.5, 3, 3.5, 4$

We return now to the deduction of further solutions of Eq. (4-1) from the integral representation contained in Eq. (4-100).

If $m = 1/q$, then Eqs. (4-102) through (4-104) become

$$\frac{1}{n} = \frac{2q(1+\lambda)}{2q-1}, \qquad (4\text{-}200)$$

$$\frac{1}{a^2} = -\frac{4\alpha q^2(1+\lambda)}{2q-1}, \qquad (4\text{-}201)$$

and

$$\nu = -\frac{(2q+1)(1+\lambda)}{2q-1}, \qquad (4\text{-}202)$$

respectively. For the case $\lambda < -1$, which will be considered below, Eqs. (4-200) through (4-202) show that $n < 0$, a is a real number, and $\nu > 0$ if $q > 0.50$.

As $n < 0$, Eqs. (4-100) and (4-101) become

$$1 - x = -a\,y_o^{1/n} \int_{\left(\frac{1}{y_o}\right)^{1/n}}^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t^{1/q} - 1}}\,dt, \qquad (4\text{-}203)$$

and

$$1 = a\,y_o^{1/n} \int_1^{\left(\frac{1}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t^{1/q} - 1}}\,dt. \qquad (4\text{-}204)$$

Consequently, when y_o is a root of Eq. (4-204), the solution of Eq. (4-1) from Eq. (4-203) is

$$x = a\,y_o^{1/n} \int_1^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t^{1/q} - 1}}\,dt. \qquad (4\text{-}205)$$

Changing the variable of integration in Eqs. (4-204) and (4-205) to $t = \tau^{2q}$ produces

$$1 = a\,y_o^{1/n}\,2q \int_1^{y_o^{-1/2qn}} \frac{\tau^{2q-1}}{\sqrt{\tau^2 - 1}}\,d\tau, \qquad (4\text{-}206)$$

and

$$x = a\,y_o^{1/n}\,2q \int_1^{\left(\frac{y_o}{y}\right)^{-1/2qn}} \frac{\tau^{2q-1}}{\sqrt{\tau^2 - 1}}\,d\tau. \qquad (4\text{-}207)$$

The results obtained by carrying out the integrals in Eqs. (4-206) and (4-207) for six values of q are as follows.

(1) If $q = 3/2$, then

$$x = \frac{3}{2} a\, y_o^{1/n}\left[\left(\frac{y_o}{y}\right)^{-1/3n}\sqrt{\left(\frac{y_o}{y}\right)^{-2/3n} - 1} + \operatorname{arccosh}\left(\frac{y_o}{y}\right)^{-1/3n}\right], \qquad (4\text{-}208)$$

in which y_o is a root of

$$\frac{2}{3a} = y_o^{1/n}\left[y_o^{-1/3n}\sqrt{y_o^{-2/3n} - 1} + \operatorname{arccosh} y_o^{-1/3n}\right], \qquad (4\text{-}209)$$

and

$$\frac{1}{n} = -\frac{3}{2}\left(|\lambda| - 1\right), \qquad (4\text{-}210)$$

$$\frac{1}{a^2} = \frac{9\alpha}{2}\left(|\lambda| - 1\right), \qquad (4\text{-}211)$$

and

$$\nu = 2\left(|\lambda| - 1\right). \qquad (4\text{-}212)$$

(2) If $q = 2$, then

$$x = \frac{4}{3} a\, y_o^{1/n}\sqrt{\left(\frac{y_o}{y}\right)^{-1/2n} - 1}\left[\left(\frac{y_o}{y}\right)^{-1/2n} + 2\right], \qquad (4\text{-}213)$$

in which y_o is a root of

$$\frac{1}{a} = \frac{4}{3} y_o^{1/n}\sqrt{y_o^{-1/2n} - 1}\,\left(y_o^{-1/2n} + 2\right), \qquad (4\text{-}214)$$

and

$$\frac{1}{n} = -\frac{4}{3}\left(|\lambda| - 1\right), \qquad (4\text{-}215)$$

$$\frac{1}{a^2} = \frac{16\alpha}{3}\left(|\lambda| - 1\right), \qquad (4\text{-}216)$$

and

$$\nu = \frac{5}{3}\left(|\lambda| - 1\right). \qquad (4\text{-}217)$$

(3) If $q = 5/2$, then

$$x = a\, y_o^{1/n}\left\{\frac{5}{8}\left(\frac{y_o}{y}\right)^{-1/5n}\sqrt{\left(\frac{y_o}{y}\right)^{-2/5n} - 1}\left[2\left(\frac{y_o}{y}\right)^{-2/5n} + 3\right] + \frac{15}{8}\operatorname{arccosh}\left(\frac{y_o}{y}\right)^{-1/5n}\right\}, \qquad (4\text{-}218)$$

in which y_o is a root of

$$\frac{1}{a} = y_o^{1/n}\left[\frac{5}{8} y_o^{-1/5n}\sqrt{y_o^{-2/5n} - 1}\,\left(2y_o^{-2/5n} + 3\right) + \frac{15}{8}\operatorname{arccosh} y_o^{-1/5n}\right], \qquad (4\text{-}219)$$

and

$$\frac{1}{n} = -\frac{5}{4}\left(|\lambda| - 1\right), \qquad (4\text{-}220)$$

$$\frac{1}{a^2} = \frac{25\alpha}{4}\left(|\lambda| - 1\right), \qquad (4\text{-}221)$$

and

$$\nu = \frac{3}{2}\left(|\lambda| - 1\right). \qquad (4\text{-}222)$$

(4) If $q = 3$, then

$$x = 6a\, y_o^{1/n}\sqrt{\left(\frac{y_o}{y}\right)^{-1/3n} - 1}\left\{\frac{1}{5}\left[\left(\frac{y_o}{y}\right)^{-1/3n} - 1\right]^2 + \frac{2}{3}\left(\frac{y_o}{y}\right)^{-1/3n} + \frac{1}{3}\right\}, \qquad (4\text{-}223)$$

in which y_o is a root of

$$\frac{1}{a} = 6\, y_o^{1/n}\sqrt{y_o^{-1/3n} - 1}\left[\frac{1}{5}\left(y_o^{-1/3n} - 1\right)^2 + \frac{2}{3} y_o^{-1/3n} + \frac{1}{3}\right], \qquad (4\text{-}224)$$

and

$$\frac{1}{n} = -\frac{6}{5}\left(|\lambda| - 1\right), \qquad (4\text{-}225)$$

$$\frac{1}{a^2} = \frac{36\alpha}{5}\left(|\lambda| - 1\right), \qquad (4\text{-}226)$$

and

$$\nu = \frac{7}{5}\left(|\lambda| - 1\right). \qquad (4\text{-}227)$$

(5) If $q = 7/2$, then

$$x = a\, y_o^{1/n}\left\{7\left(\frac{y_o}{y}\right)^{-1/7n}\sqrt{\left(\frac{y_o}{y}\right)^{-2/7n} - 1}\left[\frac{1}{6}\left(\frac{y_o}{y}\right)^{-4/7n} + \frac{5}{24}\left(\frac{y_o}{y}\right)^{-2/7n} + \frac{5}{16}\right] + \frac{35}{16}\operatorname{arccosh}\left(\frac{y_o}{y}\right)^{-1/7n}\right\}, \qquad (4\text{-}228)$$

in which y_o is a root of

$$\frac{1}{a} = y_o^{1/n}\left[7\, y_o^{-1/7n}\sqrt{y_o^{-2/7n} - 1}\left(\frac{1}{6} y_o^{-4/7n} + \frac{5}{24} y_o^{-2/7n} + \frac{5}{16}\right) + \frac{35}{16}\operatorname{arccosh} y_o^{-1/7n}\right], \qquad (4\text{-}229)$$

and

$$\frac{1}{n} = -\frac{7}{6}\left(|\lambda| - 1\right), \qquad (4\text{-}230)$$

$$\frac{1}{a^2} = \frac{49\alpha}{6}\left(|\lambda| - 1\right), \qquad (4\text{-}231)$$

and

$$\nu = \frac{4}{3}\left(|\lambda| - 1\right). \qquad (4\text{-}232)$$

(6) If $q = 4$, then

$$x = 8a\, y_o^{1/n}\sqrt{\left(\frac{y_o}{y}\right)^{-1/4n} - 1}\left\{\frac{1}{7}\left[\left(\frac{y_o}{y}\right)^{-1/4n} - 1\right]^3 + \frac{3}{5}\left[\left(\frac{y_o}{y}\right)^{-1/4n} - 1\right]^2 + \left(\frac{y_o}{y}\right)^{-1/4n}\right\}, \qquad (4\text{-}233)$$

in which y_o is a root of

$$\frac{1}{a} = 8\, y_o^{1/n}\sqrt{y_o^{-1/4n} - 1}\left[\frac{1}{7}\left(y_o^{-1/4n} - 1\right)^3 + \frac{3}{5}\left(y_o^{-1/4n} - 1\right)^2 + y_o^{-1/4n}\right], \qquad (4\text{-}234)$$

and

$$\frac{1}{n} = -\frac{8}{7}(|\lambda| - 1), \qquad (4\text{-}235)$$

$$\frac{1}{a^2} = \frac{64\alpha}{7}(|\lambda| - 1), \qquad (4\text{-}236)$$

and

$$\nu = \frac{9}{7}(|\lambda| - 1). \qquad (4\text{-}237)$$

To illustrate the determination of the value, y_o, of the solution of Eq. (4-1) at x = 0 for the six solutions of this equation written in Eqs. (4-208) through (4-237), we shall consider the q = 3/2 case, and, accordingly, Eq. (4-209). The right-hand side of Eq. (4-209) is plotted in Fig. 3 as a function of $y_o^{-1/n}$, where n is given in Eq. (4-210). The occurrence of the maximum on this curve shows that Eq. (4-209) has zero, one, or two roots, and that

$$\left(\frac{2}{3a}\right)_{max} = 0.839, \qquad (4\text{-}238)$$

together with Eq. (4-211), defines the maximum value of α, viz.,

$$\alpha_{max} = \frac{0.352}{|\lambda| - 1}, \qquad (4\text{-}239)$$

for which a solution of Eq. (4-1) will exist in the case under consideration. If $\alpha < \alpha_{max}$, Eq. (4-209) will have multiple roots, the smallest of which is of physical interest. Similar curves may be drawn for the remaining five solutions in Eqs. (4-213) through (4-237).

4.5.5 Result for m = -1

If m = -1, then Eqs. (4-102) through (4-104) become

$$\frac{1}{n} = 2(1 + \lambda), \qquad (4\text{-}240)$$

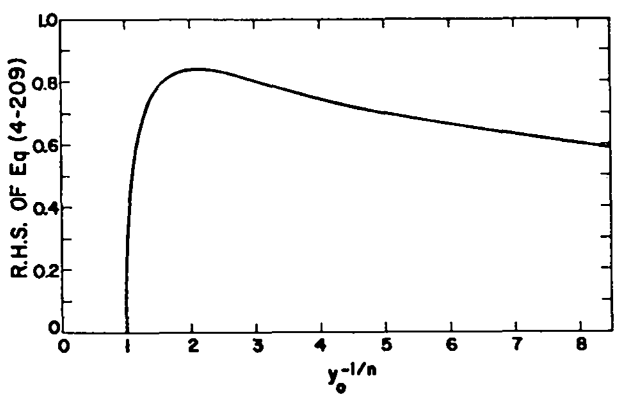

Fig. 3. Graph for determining the roots of Eq. (4-209).

$$\frac{1}{a^2} = -4\alpha(1 + \lambda), \qquad (4\text{-}241)$$

and

$$\nu = -3(1 + \lambda), \qquad (4\text{-}242)$$

respectively. Consequently, if $\lambda < -1$, then a is a real number, and $n < 0$ and $\nu > 0$. As $n < 0$, Eqs. (4-100) and (4-101) reduce to

$$\frac{1 - x}{a y_o^{1/n}} = -\int_{\left(\frac{1}{y_o}\right)^{1/n}}^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t - 1}}\, dt, \qquad (4\text{-}243)$$

and

$$1 = a y_o^{1/n} \int_1^{y_o^{-1/n}} \frac{1}{\sqrt{t - 1}}\, dt. \qquad (4\text{-}244)$$

Integrating Eq. (4-244) gives the quadratic equation

$$\left(\frac{1}{y_o}\right)^{2/n} - 4a^2\left(\frac{1}{y_o}\right)^{1/n} + 4a^2 = 0, \qquad (4\text{-}245)$$

with the solution

$$\left(\frac{1}{y_o}\right)^{1/n} = 2a^2 \pm \sqrt{4a^2(a^2 - 1)} \qquad (4\text{-}246)$$

in which n is given by Eq. (4-240), and a^2 is given by Eq. (4-241).

The proper choice of the plus or minus sign in Eq. (4-246), which provides the value of the solution of Eq. (4-1) at x = 0 in the case under consideration, may be determined by the following argument. Because $n < 0$ if $\lambda < -1$, Eq. (4-246) may be written as

$$y_o^{2(|\lambda|-1)} = 2a^2\left[1 \pm \sqrt{1 - \frac{1}{a^2}}\right]. \qquad (4\text{-}247)$$

In view of the boundary condition y(1) = 1, the solution of Eq. (4-1) in the limit as α approaches zero is y(x) = 1. Accordingly, in the limit as $\alpha \rightarrow 0$, we must obtain the solution $y_o = 1$ from Eq. (4-247). From Eq. (4-241), we see that

$$\frac{1}{a^2} \xrightarrow[\alpha\to 0]{} 0, \qquad (4\text{-}248)$$

so that we may expand the square root in Eq. (4-247) to obtain

$$y_o^{2(|\lambda|-1)} = 2a^2\left[1 \pm \left(1 - \frac{1}{2a^2} - \frac{1}{8a^4} - \frac{1}{16a^6} \cdots\right)\right]. \qquad (4\text{-}249)$$

If we should choose the plus sign in Eq. (4-249), we would get

$$y_o^{2(|\lambda|-1)} \xrightarrow[\alpha\to 0]{} \infty, \qquad (4\text{-}250)$$

which is clearly unacceptable. On the other hand, using the minus sign in Eq. (4-249) gives

$$y_o^{2(|\lambda|-1)} \xrightarrow[\alpha\to 0]{} 1, \qquad (4\text{-}251)$$

as required. Therefore, the value, y_o, of the solution of Eq. (4-1) at x = 0 is given by the relation

$$y_o^{2(|\lambda|-1)} = 2a^2\left[1 - \sqrt{1 - \frac{1}{a^2}}\right], \qquad (4\text{-}252)$$

for the current case.

Moreover, Eq. (4-252) indicates that the inequality

$$\frac{1}{a^2} \leq 1 \qquad (4\text{-}253)$$

must be satisfied if the solution for y_o is going to be a real number. Combining Eqs. (4-253) and (4-241) with $\lambda < -1$ yields

$$\alpha \leq \alpha_{max} = \frac{1}{4(|\lambda| - 1)} \qquad (4\text{-}254)$$

as the condition that must be satisfied by α for a solution of Eq. (4-1) to exist when $\nu = 3(|\lambda|-1)$.

From Eqs. (4-243) and (4-244) it is found that

$$x = ay_o^{1/n} \int_1^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t - 1}}\, dt, \qquad (4\text{-}255)$$

which, upon integration, becomes

$$y^{1/n} = y_o^{1/n} + \frac{x^2}{4a^2} y_o^{-1/n}. \qquad (4\text{-}256)$$

Substituting Eqs. (4-240) and (4-252) into Eq. (4-256) and simplifying the result produces

$$y^{2(|\lambda|-1)} = \frac{2a^2\left(1 - \sqrt{1 - \frac{1}{a^2}}\right)}{1 + a^2x^2\left[-\frac{1}{a^2} + 2\left(1 - \sqrt{1 - \frac{1}{a^2}}\right)\right]}, \qquad (4\text{-}257)$$

in which a^2 is given by Eq. (4-241). The result found in Eq. (4-257) is the solution of Eq. (4-1) subject to the conditions of Eqs. (4-242) and (4-254) and the boundary conditions $y'(0) = 0$ and $y(1) = 1$ if $\lambda < -1$. When $\alpha = \alpha_{max}$, then a = 1, and Eq. (4-257) reduces to

$$y = \left(\frac{2}{1 + x^2}\right)^{\frac{1}{2(|\lambda|-1)}}. \qquad (4\text{-}258)$$

4.5.6 Result for m = -2

If m = -2, it follows from Eq. (4-104) that $\lambda = -1$ for all values of ν. If we assume that $\nu > 0$, then $n < 0$, because Eq. (4-102) becomes

$$\frac{1}{n} = -\frac{\nu}{2}. \qquad (4\text{-}259)$$

Also, in this case Eq. (4-103) gives

$$a^2 = \frac{2}{\alpha\nu}. \qquad (4\text{-}260)$$

Now, because $n < 0$, Eqs. (4-101) and (4-259) combine into

$$\frac{1}{a} y_o^{\nu/2} = \int_1^{y_o^{\nu/2}} \frac{1}{\sqrt{t^2 - 1}}\, dt, \qquad (4\text{-}261)$$

which integrates out to produce

$$y_o^{\nu/2} = \cosh\left(\sqrt{\frac{\alpha\nu}{2}}\, y_o^{\nu/2}\right) \qquad (4\text{-}262)$$

as the transcendental equation for the value, y_o, of the solution of Eq. (4-1) at x = 0 for this case. With the definition

$$A_o \equiv \sqrt{\frac{\alpha\nu}{2}}\, y_o^{\nu/2}, \qquad (4\text{-}263)$$

Eq. (4-262) may be written as

$$\sqrt{\frac{2}{\alpha\nu}} = \frac{\cosh A_o}{A_o}. \qquad (4\text{-}264)$$

The right-hand side of Eq. (4-264) is plotted in Fig. 4 as a function of A_o. The occurrence of the minimum point on the curve in this figure defines the maximum allowed value of α such that a solution of Eq. (4-1) exists in the current case, that is,

$$\alpha_{max} = \frac{2}{\nu(1.50)^2} = \frac{0.89}{\nu}. \qquad (4\text{-}265)$$

If $\alpha < \alpha_{max}$, then Eq. (4-264) has two roots, of which the smaller is of physical interest because $y_o \to 1$ as $\alpha \to 0$.

The explicit solution of Eq. (4-1) is found from Eq. (4-100), which now reduces to

$$\frac{(1-x)}{a} y_o^{\nu/2} = \int_1^{y_o^{\nu/2}} \frac{1}{\sqrt{t^2 - 1}}\, dt - \int_1^{\left(\frac{y_o}{y}\right)^{\nu/2}} \frac{1}{\sqrt{t^2 - 1}}\, dt, \qquad (4\text{-}266)$$

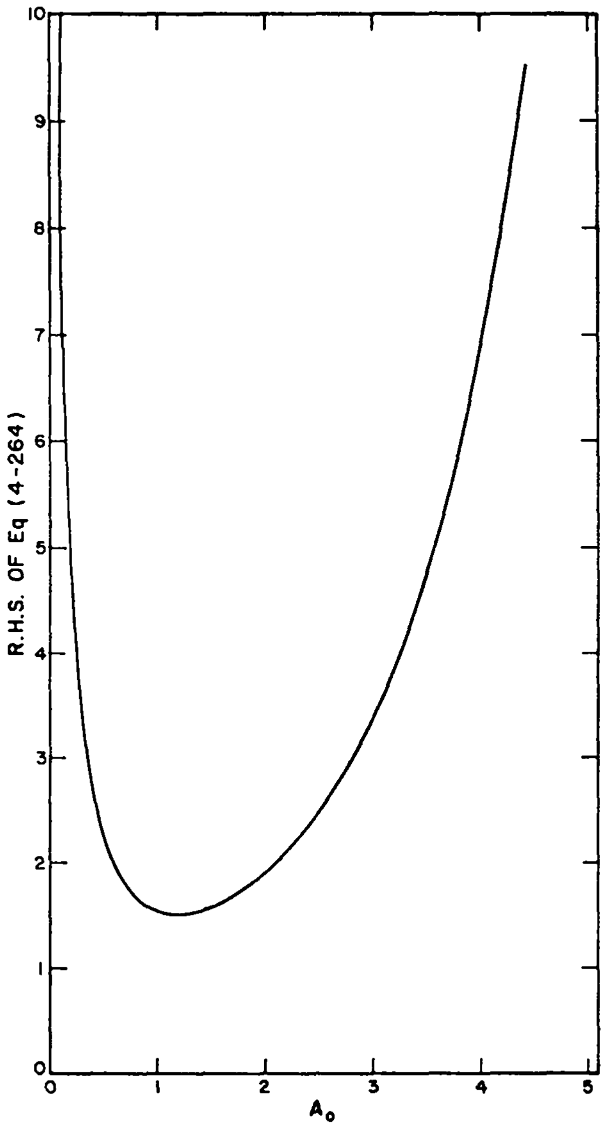

Fig. 4. Graph for determining the roots of Eq. (4-264).

or, in view of Eq. (4-261),

$$x\, y_o^{\nu/2} = a \int_1^{\left(\frac{y_o}{y}\right)^{\nu/2}} \frac{1}{\sqrt{t^2 - 1}}\, dt. \qquad (4\text{-}267)$$

Integrating in Eq. (4-267) results in

$$y = \frac{y_o}{\cosh^{2/\nu}\left(\sqrt{\frac{a\nu}{2}}\, x\, y_o^{\nu/2}\right)}, \qquad (4\text{-}268)$$

in which y_o is a root of Eq. (4-264). When $\lambda = -1$, Eq. (4-268) is the solution of Eq. (4-1) provided that $\nu > 0$ and $\alpha \leq \alpha_{max}$, where α_{max} is given in Eq. (4-265). The result of Eq. (4-268) is not valid if $\nu = 0$.

We now turn to the $\lambda = -1$ and $\nu = 0$ case.

4.5.7 Result for $\nu = 0$ and $\lambda = -1$

If $\nu = 0$ and $\lambda = -1$, then Eq. (4-1) assumes the form

$$yy'' - (y')^2 + \alpha y^2 = 0. \qquad (4\text{-}269)$$

This nonlinear differential equation is invariant under the two-parameter Lie group generated by the two infinitesimal transformation symbols

$$U_1 f = \frac{\partial f}{\partial x}, \qquad (4\text{-}270)$$

and

$$U_2 f = y\,\frac{\partial f}{\partial y}. \qquad (4\text{-}271)$$

Moreover, this two-parameter group is of the first type with the canonical variables

$$X = x, \qquad (4\text{-}272)$$

and

$$Y = \ln y. \qquad (4\text{-}273)$$

Hence

$$y = \exp\,(Y), \qquad (4\text{-}274)$$

$$y' = \exp\,(Y)\,\frac{dY}{dX}, \qquad (4\text{-}275)$$

and

$$y'' = \exp\,(Y)\left[\frac{d^2Y}{dX^2} + \left(\frac{dY}{dX}\right)^2\right], \qquad (4\text{-}276)$$

and the canonical form of Eq. (4-269) is found to be

$$\frac{d^2Y}{dX^2} = -\,\alpha, \qquad (4\text{-}277)$$

from which two quadratures produce

$$Y = -\frac{\alpha}{2}X^2 + C_1X + C_2, \qquad (4\text{-}278)$$

where C_1 and C_2 are arbitrary constants. Upon reverting back to the original variables, Eq. (4-278) becomes

$$y = \exp\left(-\frac{\alpha}{2}x^2 + C_1x + C_2\right), \qquad (4\text{-}279)$$

which is the general solution of Eq. (4-269). If we now impose the boundary conditions, $y'(0) = 0$ and $y(1) = 1$, Eq. (4-279) reduces to

$$y = \exp\left[\frac{\alpha}{2}(1 - x^2)\right], \qquad (4\text{-}280)$$

and this is the solution of Eq. (4-1) when $\nu = 0$ and $\lambda = -1$ for the stated boundary conditions. There is no restriction on the allowed values of α in this result.

4.5.8 Result for m = -3

In this and future subsections, solutions of Eq. (4-1), which may be expressed in terms of elliptic functions, will be derived.

If m = -3, then Eqs. (4-102) through (4-104) become

$$\frac{1}{n} = -2(1+\lambda), \qquad (4\text{-}281)$$

$$\frac{1}{a^2} = \frac{4\alpha}{3}(1+\lambda), \qquad (4\text{-}282)$$

and

$$\nu = 5(1+\lambda), \qquad (4\text{-}283)$$

respectively. Accordingly, if $\lambda > -1$, then $n < 0$ and a is a real number, and in this case Eqs. (4-101) and (4-100) reduce to the forms

$$1 = a\, y_o^{1/n} \int_1^{\left(\frac{1}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t^3-1}}\, dt, \qquad (4\text{-}284)$$

and

$$1 - x = a\, y_o^{1/n} \int_{\left(\frac{y}{y_o}\right)^{1/n}}^{\left(\frac{1}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t^3-1}}\, dt. \qquad (4\text{-}285)$$

Because of Eq. (4-284), we may write Eq. (4-285) as

$$x = a\, y_o^{1/n} \int_1^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t^3-1}}\, dt. \qquad (4\text{-}286)$$

Now from Eq. (4-286), the solution of Eq. (4-1) may be expressed in terms of the Jacobian elliptic function, $\mathrm{cn}(u|k)$. With the value of n given in Eq. (4-281), the integration of Eq. (4-286) provides

$$\frac{x}{a}\, y_o^{2(1+\lambda)} = \frac{1}{3^{1/4}}\, \mathrm{cn}^{-1}(\cos\phi|k), \qquad (4\text{-}287)$$

in which the modulus, k, has the value,

$$k = \sin(\pi/12), \qquad (4\text{-}288)$$

and

$$\cos\phi = \frac{\sqrt{3} + 1 - \left(\frac{y_o}{y}\right)^{2(1+\lambda)}}{\sqrt{3} - 1 + \left(\frac{y_o}{y}\right)^{2(1+\lambda)}}. \qquad (4\text{-}289)$$

Accordingly, if $\lambda > -1$ and Eq. (4-283) holds, then the solution of Eq. (4-1) found from Eq. (4-287) is

$$\frac{\sqrt{3} + 1 - \left(\frac{y_o}{y}\right)^{2(1+\lambda)}}{\sqrt{3} - 1 + \left(\frac{y_o}{y}\right)^{2(1+\lambda)}} = \mathrm{cn}\left(3^{1/4} y_o^{2(1+\lambda)}\, \frac{x}{a}\Big|k\right), \qquad (4\text{-}290)$$

in which a is given by Eq. (4-282).

The value, y_o, of the solution at $x = 0$, which appears in Eq. (4-290), is a root of Eq. (4-284) that may be written in terms of $F(\phi_o|k)$, the incomplete elliptic integral of the first kind. That is, if we set

$$X_o \equiv y_o^{2(1+\lambda)}, \qquad (4\text{-}291)$$

then Eq. (4-284) may be expressed as

$$\frac{3^{1/4}}{a} = \frac{F(\phi_o|k)}{X_o}, \qquad (4\text{-}292)$$

wherein

$$\cos\phi_o = \frac{\sqrt{3} + 1 - X_o}{\sqrt{3} - 1 + X_o}. \qquad (4\text{-}293)$$

The right-hand side of Eq. (4-292) is plotted in Fig. 5 as a function of X_o. The existence of a maximum on this curve together with Eq. (4-282) shows that the maximum allowed value of α for a given $\lambda > -1$ is found from

$$3^{1/4}\sqrt{\frac{4}{3}(1+\lambda)\,\alpha_{max}} = 0.675, \qquad (4\text{-}294)$$

which simplifies to

$$\alpha_{max} = \frac{0.198}{1+\lambda}. \qquad (4\text{-}295)$$

If $\alpha > \alpha_{max}$, then Eq. (4-292) has no roots, and, correspondingly, Eq. (4-1) has no solution for the case at hand. When $\alpha = \alpha_{max}$, there is one root, and when $\alpha < \alpha_{max}$, the smaller of the two roots is the one of physical interest.

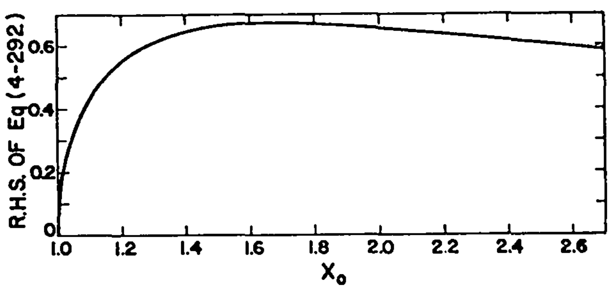

Fig. 5. Graph for determining the roots of Eq. (4-292).

4.5.9 Result for m = -4

In the m = -4 case, if $\lambda > -1$, then $n < 0$ and $\nu > 0$, as Eqs. (4-102) through (4-104) reduce to

$$\frac{1}{n} = -(1 + \lambda), \tag{4-296}$$

$$\frac{1}{a^2} = \frac{\alpha}{2}(1 + \lambda), \tag{4-297}$$

and

$$\nu = 3(1 + \lambda), \tag{4-298}$$

respectively, while Eqs. (4-101) and (4-100) become

$$y_o^{1+\lambda} = a \int_1^{y_o^{1+\lambda}} \frac{1}{\sqrt{t^4 - 1}}\, dt, \tag{4-299}$$

and

$$x\, y_o^{1+\lambda} = a \int_1^{\left(\frac{y_o}{y}\right)^{1+\lambda}} \frac{1}{\sqrt{t^4 - 1}}\, dt. \tag{4-300}$$

From Eq. (4-300) we obtain

$$x\, y_o^{1+\lambda} = \frac{a}{\sqrt{2}}\, \mathrm{cn}^{-1}(\cos\phi|k) = \frac{a}{\sqrt{2}} F(\phi|k), \tag{4-301}$$

where

$$k = \sin(\pi/4), \tag{4-302}$$

and

$$\cos\phi = \left(\frac{y}{y_o}\right)^{1+\lambda}. \tag{4-303}$$

These last three relations are the solution of Eq. (4-1) when $\lambda > -1$, and Eq. (4-298) holds. Using Eq. (4-297) allows this solution to be written in the explicit form

$$y = y_o\left\{\mathrm{cn}\left[\sqrt{\alpha(1+\lambda)}\; x\, y_o^{1+\lambda}\right]\right\}^{\frac{1}{1+\lambda}}, \tag{4-304}$$

in which the modulus of the Jacobian elliptic function is that of Eq. (4-302).

The value of the solution at x = 0 is a root of Eq. (4-299) that reduces to the following relation involving an incomplete elliptic integral of the first kind;

$$y_o^{1+\lambda} = \frac{a}{\sqrt{2}} F\left(\phi_o\middle|\frac{1}{\sqrt{2}}\right), \tag{4-305}$$

or, with Eq. (4-297), to

$$\sqrt{\alpha(1+\lambda)} = \frac{F\left(\phi_o\middle|\frac{1}{\sqrt{2}}\right)}{y_o^{1+\lambda}}, \tag{4-306}$$

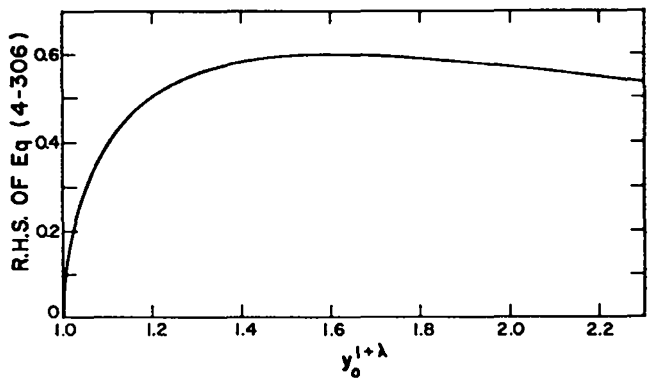

Fig. 6. Graph for determining the roots of Eq. (4-306).

wherein

$$\cos\phi_o = \frac{1}{y_o^{1+\lambda}}. \tag{4-307}$$

The right-hand side of the transcendental equation for y_o contained in Eq. (4-306) is plotted in Fig. 6 as a function of $y_o^{1+\lambda}$. The maximum point on the curve in this figure leads to a maximum allowed value of α given by

$$\alpha_{max} = \frac{0.354}{1 + \lambda}. \tag{4-308}$$

If $\alpha = \alpha_{max}$, then Eq. (4-306) has one root, and when $\alpha < \alpha_{max}$, there are two roots of which the smaller is of physical interest, because the corresponding value found for y_o is less than the value of y_o for $\alpha = \alpha_{max}$. If $\alpha > \alpha_{max}$, then Eq. (4-1) has no solution under the conditions of the current case.

4.5.10 Result for m = -6

If we let m = -6, then Eq. (4-104) becomes

$$\nu = 2(1 + \lambda), \tag{4-309}$$

so that $\nu > 0$ if $\lambda > -1$. Since $n < 0$ in this case, as Eq. (4-102) reduces to

$$\frac{1}{n} = -\frac{(1 + \lambda)}{2}, \tag{4-310}$$

the value of the solution of Eq. (4-1) at x = 0 will be a root of the corresponding form of Eq. (4-101), viz.,

$$y_o^{\frac{1+\lambda}{2}} = a \int_1^{y_o^{\frac{1+\lambda}{2}}} \frac{1}{\sqrt{t^6 - 1}}\, dt, \tag{4-311}$$

in which, from Eq. (4-103),

$$\frac{1}{a^2} = \frac{\alpha}{6}(1 + \lambda), \tag{4-312}$$

so a is a real number if $\lambda > -1$. Now let

$$L_o \equiv y_o^{\frac{1+\lambda}{2}}, \tag{4-313}$$

in Eq. (4-311), so that

$$\frac{L_o}{a} = \int_1^\infty \frac{1}{\sqrt{t^6 - 1}}\,dt - \int_{L_o}^\infty \frac{1}{\sqrt{t^6 - 1}}\,dt. \tag{4-314}$$

These hyperelliptic integrals may be evaluated in terms of the incomplete elliptic integral of the first kind. We find that

$$\int_1^\infty \frac{1}{\sqrt{t^6 - 1}}\,dt = \frac{1}{2\cdot 3^{1/4}}\left[F(\psi_1|k) - F(\phi_1|k)\right], \tag{4-315}$$

in which the modulus is given by

$$k^2 = \frac{2 + \sqrt{3}}{4} = \sin^2\left(\frac{5\pi}{12}\right), \tag{4-316}$$

and

$$\cos\psi_1 = 2 - \sqrt{3}, \tag{4-317}$$

together with

$$\cos\phi_1 = \frac{(\sqrt{3} - 1) + 1}{(\sqrt{3} + 1) - 1} = 1. \tag{4-318}$$

This last relation implies that $\phi_1 = 0$, so Eq. (4-315) simplifies to

$$\int_1^\infty \frac{1}{\sqrt{t^6 - 1}}\,dt = \frac{1}{2\cdot 3^{1/4}}\,F(\psi_1|k). \tag{4-319}$$

Since the second integral in Eq. (4-314) is

$$\int_{L_o}^\infty \frac{1}{\sqrt{t^6 - 1}}\,dt = \frac{1}{2\cdot 3^{1/4}}\left[F(\psi_1|k) - F(\phi_o|k)\right], \tag{4-320}$$

with the same modulus as Eq. (4-316), and

$$\cos\psi_o = \frac{(\sqrt{3} - 1)\,L_o^2 + 1}{(\sqrt{3} + 1)\,L_o^2 - 1}, \tag{4-321}$$

it is found that Eq. (4-314) may be written in the final form

$$\frac{2\cdot 3^{1/4}}{a} = \frac{F(\psi_o|k)}{L_o}. \tag{4-322}$$

The right-hand side of Eq. (4-322) is plotted

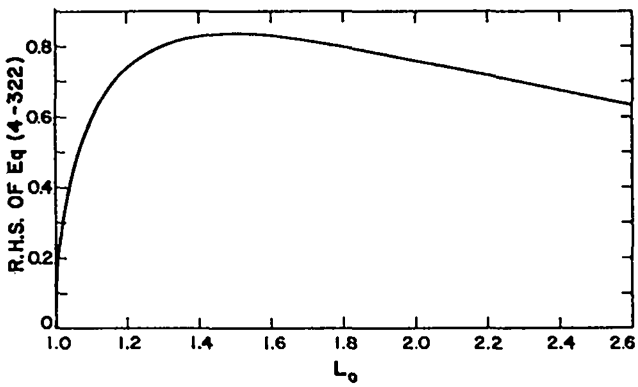

Fig. 7. Graph for determining the roots of Eq. (4-322).

in Fig. 7 as a function of L_o. From the maximum point on this curve and Eq. (4-312), the maximum allowed value of α for the current case comes out of

$$2\cdot 3^{1/4}\sqrt{\frac{\alpha_{max}(1 + \lambda)}{6}} = 0.833 \tag{4-323}$$

with the value

$$\alpha_{max} = \frac{0.601}{1 + \lambda}. \tag{4-324}$$

If $\alpha > \alpha_{max}$, then Eq. (4-322) has no roots; if $\alpha = \alpha_{max}$, one root; and if $\alpha < \alpha_{max}$, two roots, the smaller of which is of interest.

When Eq. (4-322) has a root, then the solution of Eq. (4-1) deduced from Eq. (4-100) assumes the form

$$\frac{1 - x}{a y_o^{1/n}} = \int_{\left(\frac{1}{y_o}\right)^{1/n}}^{1} \frac{1}{\sqrt{t^6 - 1}}\,dt - \int_1^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t^6 - 1}}\,dt, \tag{4-325}$$

so that, in view of Eq. (4-311) for y_o,

$$\frac{x}{a y_o^{1/n}} = \int_1^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{1}{\sqrt{t^6 - 1}}\,dt. \tag{4-326}$$

Therefore,

$$\frac{x}{a y_o^{1/n}} = \int_1^\infty \frac{1}{\sqrt{t^6 - 1}}\,dt - \int_{\left(\frac{y}{y_o}\right)^{1/n}}^\infty \frac{1}{\sqrt{t^6 - 1}}\,dt, \tag{4-327a}$$

$$= \frac{1}{2\cdot 3^{1/4}}\Big[F(\psi_1|k) - F(\phi_1|k) - F(\psi_1|k) + F(\psi|k)\Big], \qquad (4\text{-}327b)$$

where

$$\cos\psi = \frac{(\sqrt{3}-1)\left(\frac{y}{y_o}\right)^{2/n} + 1}{(\sqrt{3}+1)\left(\frac{y}{y_o}\right)^{2/n} - 1}. \qquad (4\text{-}328)$$

Consequently, it has been found that the solution of Eq. (4-1) is

$$x = \frac{a y_o^{1/n}}{2\cdot 3^{1/4}} F(\psi|k), \qquad (4\text{-}329)$$

with the modulus of Eq. (4-316), the value of a from Eq. (4-312), and the value of y_o from Eq. (4-322), when Eq. (4-309) holds, and $\lambda > -1$. In terms of the Jacobian elliptic function, the solution in Eq. (4-329) becomes

$$\frac{(\sqrt{3}-1)\left(\frac{y_o}{y}\right)^{1+\lambda} + 1}{(\sqrt{3}+1)\left(\frac{y_o}{y}\right)^{1+\lambda} - 1} = \mathrm{cn}\left(\frac{2\cdot 3^{1/4}x}{a y_o^{1/n}}\middle| k\right). \qquad (4\text{-}330)$$

4.5.11 Result for m = 6

If m = 6, Eq. (4-101) is

$$\frac{1}{a} = y_o^{1/n}\int_{\left(\frac{1}{y_o}\right)^{1/n}}^{1} \frac{t^3}{\sqrt{1-t^6}}\,dt, \qquad (4\text{-}331)$$

and Eq. (4-100) is

$$\frac{1-x}{a} = y_o^{1/n}\int_{\left(\frac{1}{y_o}\right)^{1/n}}^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{t^3}{\sqrt{1-t^6}}\,dt. \qquad (4\text{-}332)$$

Because Eqs. (4-102) through (4-104) are, if m = 6,

$$\frac{1}{n} = \frac{1+\lambda}{4}, \qquad (4\text{-}333)$$

$$\frac{1}{a^2} = \frac{\alpha(1+\lambda)}{12}, \qquad (4\text{-}334)$$

and

$$\nu = \frac{1+\lambda}{2}, \qquad (4\text{-}335)$$

respectively, the plus sign has been used to write out Eqs. (4-331) and (4-332) as $n > 0$ and $\nu > 0$ if $\lambda > -1$ in the case. The integrals appearing in Eqs. (4-331) and (4-332) may be put into easily computable forms with the formula

$$\int \frac{t^3}{\sqrt{1-t^6}}\,dt = \frac{-\sqrt{1-t^6}}{t^2 - 1 - \sqrt{3}} - \frac{(\sqrt{3}-1)}{2\cdot 3^{1/4}} F(\psi|k) + 3^{1/4} E(\psi|k), \qquad (4\text{-}336)$$

in which E is an incomplete elliptic integral of the second kind with the indicated argument and modulus; the modulus is

$$k^2 = \frac{2+\sqrt{3}}{4} = \sin^2\left(\frac{5\pi}{12}\right), \qquad (4\text{-}337)$$

and

$$\cos\psi = \frac{t^2 - 1 + \sqrt{3}}{t^2 - 1 - \sqrt{3}}, \text{ for } 0 \le \psi \le \pi. \qquad (4\text{-}338)$$

Through the utilization of Eq. (4-336), it is found that Eq. (4-331) provides

$$\frac{1}{a y_o^{1/n}} = \frac{(\sqrt{3}-1)}{2\cdot 3^{1/4}}\Big[F(\psi_o|k) - F(\psi_1|k)\Big] + 3^{1/4}\Big[E(\psi_1|k) - E(\psi_o|k)\Big] + \frac{\sqrt{1 - \left(\frac{1}{y_o^{1/n}}\right)^6}}{\left(\frac{1}{y_o^{1/n}}\right)^2 - 1 - \sqrt{3}}, \qquad (4\text{-}339)$$

wherein,

$$\cos\psi_1 = -1 \qquad (4\text{-}340)$$

and

$$\cos\psi_o = \frac{\left(\frac{1}{y_o^{1/n}}\right)^2 - 1 + \sqrt{3}}{\left(\frac{1}{y_o^{1/n}}\right)^2 - 1 - \sqrt{3}}, \qquad (4\text{-}341)$$

as the transcendental equation for the value, y_o, of the solution of Eq. (4-1) at x = 0. Because the range of the arguments of elliptic integrals may be extended by the formulae,

$$E(m\pi \pm \phi|k) = 2mE \pm E(\phi|k), \qquad (4\text{-}342)$$

where E is the complete elliptic integral of the second kind, and

$$F(m\pi \pm \phi|k) = 2mK \pm F(\phi|k), \qquad (4\text{-}343)$$

where K is the complete elliptical integral of the first kind, we have for use in simplifying Eq. (4-339),

$$F(\psi_1|k) = F(\pi - \phi_1|k) = 2K - F(\phi_1|k) = 2K, \qquad (4\text{-}344)$$

$$E(\psi_1|k) = E(\pi - \phi_1|k) = 2E - E(\phi_1|k) = 2E, \qquad (4\text{-}345)$$

$$F(\psi_o|k) = F(\pi - \phi_o|k) = 2K - F(\phi_o|k), \qquad (4\text{-}346)$$

and

$$E(\psi_o|k) = E(\pi - \phi_o|k) = 2E - E(\phi_o|k). \qquad (4\text{-}347)$$

In Eqs. (4-344) and (4-345), $\phi_1 = 0$, and in Eqs. (4-346) and (4-347)

$$\cos \phi_o = \frac{1 - \sqrt{3} - \left(\frac{1}{y_o^{1/n}}\right)^2}{1 - \sqrt{3} + \left(\frac{1}{y_o^{1/n}}\right)^2}. \qquad (4\text{-}348)$$

Upon introducing Eqs. (4-344) through (4-347) into Eq. (4-339), we obtain the relation

$$\frac{1}{a} = y_o^{1/n}\left[\frac{\sqrt{1 - \left(\frac{1}{y_o^{1/n}}\right)^6}}{\left(\frac{1}{y_o^{1/n}}\right)^2 - 1 - \sqrt{3}} + 3^{1/4}\, E(\phi_o|k) - \frac{(\sqrt{3} - 1)}{2 \cdot 3^{1/4}} F(\phi_o|k)\right]. \qquad (4\text{-}349)$$

The right-hand side of Eq. (4-349) is plotted in Fig. 8 as a function of $y_o^{1/n}$. The curve is seen to be a monotonically increasing function, so that Eq. (4-349) has only a single root for y_o, the value of the solution of Eq. (4-1) at x = 0 for each value of α. This may be contrasted with previous cases in which α was restricted by the inequality, $\alpha \le \alpha_{max}$.

When y_o is the root of Eq. (4-349), it is found that Eq. (4-332) reduces to

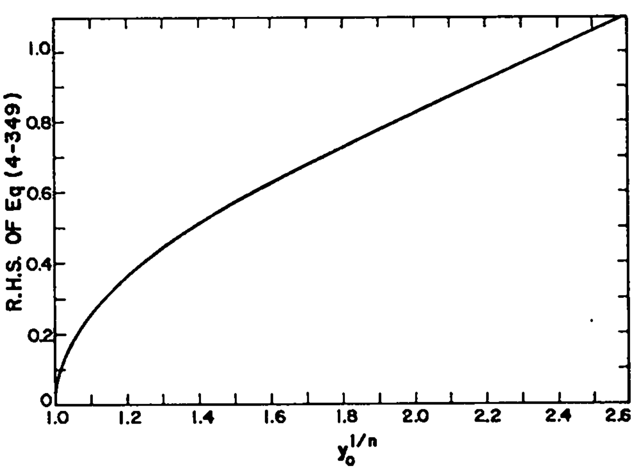

Fig. 8. Graph for determining the root of Eq. (4-349).

$$x = a\, y_o^{1/n} \int_{\left(\frac{y}{y_o}\right)^{1/n}}^{1} \frac{t^3}{\sqrt{1 - t^6}}\, dt. \qquad (4\text{-}350)$$

With Eq. (4-336), this first becomes

$$\frac{x}{a y_o^{1/n}} = \frac{(\sqrt{3} - 1)}{2 \cdot 3^{1/4}} [F(\psi|k) - F(\psi_1|k)] + 3^{1/4} [E(\psi_1|k) - E(\psi|k)] + \frac{\sqrt{1 - \left(\frac{y}{y_o}\right)^{6/n}}}{\left(\frac{y}{y_o}\right)^{2/n} - 1 - \sqrt{3}}, \qquad (4\text{-}351)$$

in which $\psi_1 = \pi$, k is the modulus given in Eq. (4-337), and

$$\cos \psi = \frac{\left(\frac{y}{y_o}\right)^{2/n} - 1 + \sqrt{3}}{\left(\frac{y}{y_o}\right)^{2/n} - 1 - \sqrt{3}}. \qquad (4\text{-}352)$$

Now let $\psi = \pi - \phi$ and note that

$$F(\psi_1|k) = 2K, \qquad (4\text{-}353)$$

$$E(\psi_1|k) = 2E, \qquad (4\text{-}354)$$

$$F(\psi|k) = F(\pi - \phi|k) = 2K - F(\phi|k), \qquad (4\text{-}355)$$

and

$$E(\psi|k) = E(\pi - \phi|k) = 2E - E(\phi|k). \qquad (4\text{-}356)$$

Then, it may be observed that Eq. (4-351) simplifies to

$$\frac{x}{a y_o^{1/n}} = \frac{\sqrt{1 - \left(\frac{y}{y_o}\right)^{6/n}}}{\left(\frac{y}{y_o}\right)^{2/n} - 1 - \sqrt{3}} + 3^{1/4}\, E(\phi|k) - \frac{(\sqrt{3} - 1)}{2 \cdot 3^{1/4}} F(\phi|k), \qquad (4\text{-}357)$$

wherein

$$\cos \phi = \frac{(\sqrt{3} - 1)\left(\frac{y_o}{y}\right)^{2/n} + 1}{(\sqrt{3} + 1)\left(\frac{y_o}{y}\right)^{2/n} - 1}. \qquad (4\text{-}358)$$

To sum up, Eq. (4-357) is the solution of Eq. (4-1) when $\nu = (1+\lambda)/2$, $\lambda > -1$, $0 \le x \le 1$, $1 \le y \le y_o$, $y'(0) = 0$, $y(1) = 1$, the value of n is from Eq. (4-333), the value of a is from Eq. (4-334) and y_o is the root of Eq. (4-349).

4.5.12 Result for m = 4

When m = 4, the following relations are valid:

$$\frac{1}{n} = \frac{1 + \lambda}{3}, \qquad (4\text{-}359)$$

$$\frac{1}{a^2} = \frac{\alpha}{6}(1 + \lambda), \qquad (4\text{-}360)$$

and

$$\nu = \frac{1 + \lambda}{3}. \qquad (4\text{-}361)$$

Under the assumption that $\lambda > -1$, so that $\nu > 0$, $n > 0$, and a is a real number, Eq. (4-101) now provides

$$\frac{1}{a} = y_o^{1/n} \int_{\left(\frac{1}{y_o}\right)^{1/n}}^{1} \frac{t^2}{\sqrt{1 - t^4}}\, dt. \qquad (4\text{-}362)$$

The transcendental equation for the value, y_o, of the solution of Eq. (4-1) at x = 0 which comes out of Eq. (4-362) is found to be

$$\frac{1}{a\sqrt{2}} = y_o^{1/n}\left[E(\phi_o|k) - \frac{1}{2}F(\phi_o|k)\right], \qquad (4\text{-}363)$$

in which

$$\cos\phi_o = y_o^{-1/n}, \qquad (4\text{-}364)$$

and the modulus of the elliptic integrals is given by

$$k = \sin(\pi/4). \qquad (4\text{-}365)$$

The right-hand side of Eq. (4-363) is plotted in Fig. 9 as a function of $y_o^{1/n}$, and the curve is seen

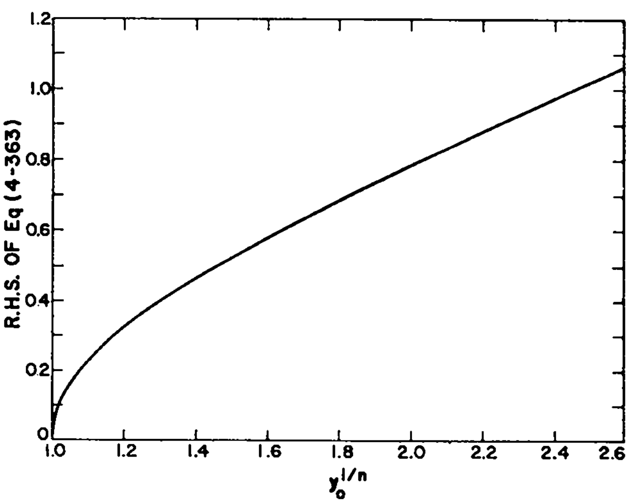

Fig. 9. Graph for determining the root of Eq. (4-363).

to be monotonically increasing. This fact indicates that Eq. (4-363) has only a single root for each value of α, so that all values of α are permitted.

When y_o is the root of Eq. (4-363), the solution of Eq. (4-1) that arises from Eq. (4-100) is

$$x = a\, y_o^{1/n} \int_{\left(\frac{y}{y_o}\right)^{1/n}}^{1} \frac{t^2}{\sqrt{1 - t^4}}\, dt, \qquad (4\text{-}366)$$

which reduces directly into the form,

$$x = a\, y_o^{1/n} \sqrt{2}\,\left[E(\phi|k) - \frac{1}{2}F(\phi|k)\right], \qquad (4\text{-}367)$$

with the modulus given in Eq. (4-365) and

$$\cos\phi = \left(\frac{y}{y_o}\right)^{1/n}. \qquad (4\text{-}368)$$

The result contained in Eq. (4-367) is the solution of Eq. (4-1) subject to the above stated conditions of the present case.

4.5.13 Result for m = 3

If m = 3, then

$$\frac{1}{n} = \frac{2}{5}(1 + \lambda), \qquad (4\text{-}369)$$

$$\frac{1}{a^2} = \frac{4\alpha}{15}(1 + \lambda), \qquad (4\text{-}370)$$

and

$$\nu = \frac{1 + \lambda}{5}. \qquad (4\text{-}371)$$

When $\lambda > -1$, then $n > 0$ and $\nu > 0$, and Eqs. (4-101) and (4-100) yield

$$\frac{1}{a} = y_o^{1/n} \int_{\left(\frac{1}{y_o}\right)^{1/n}}^{1} \sqrt{\frac{t^3}{1 - t^3}}\, dt, \qquad (4\text{-}372)$$

as the transcendental equation for y_o, and

$$x = a\, y_o^{1/n} \int_{\left(\frac{y}{y_o}\right)^{1/n}}^{1} \sqrt{\frac{t^3}{1 - t^3}}\, dt, \qquad (4\text{-}373)$$

as the desired solution of Eq. (4-1). Let $t = \tau^2$ in both of these last two relations, so that they become

$$\frac{1}{a} = 2\, y_o^{1/n} \int_{\left(\frac{1}{y_o}\right)^{1/2n}}^{1} \frac{\tau^4}{\sqrt{1 - \tau^6}}\, d\tau, \qquad (4\text{-}374)$$

and

$$x = 2a\, y_o^{1/n} \int_{\left(\frac{y}{y_o}\right)^{1/2n}}^{1} \frac{\tau^4}{\sqrt{1 - \tau^6}}\, d\tau. \qquad (4\text{-}375)$$

The reduction of Eqs. (4-374) and (4-375) to easily computable forms may be done through the utilization of the formula

$$\int \frac{\tau^4}{\sqrt{1 - \tau^6}}\, d\tau = \frac{(1 - \sqrt{3})\tau\sqrt{1 - \tau^6}}{2[(\sqrt{3} - 1)\tau^2 + 1]} + \frac{(1 + \sqrt{3})}{4 \cdot 3^{1/4}} F(\psi|k) - \frac{3^{1/4}}{2} E(\psi|k), \qquad (4\text{-}376)$$

in which the elliptic integrals have the modulus,

$$k = \sin(\pi/12), \qquad (4\text{-}377)$$

and

$$\cos\psi = \frac{(1 + \sqrt{3})\tau^2 - 1}{(\sqrt{3} - 1)\tau^2 + 1}, \text{ for } 0 \le \psi \le \pi. \qquad (4\text{-}378)$$

With Eqs. (4-376) and (4-378), the result that comes out of Eq. (4-374) is found to be

$$\frac{1}{a} = y_o^{1/n}\left[\frac{(\sqrt{3} - 1)\, y_o^{-1/2n}\sqrt{1 - y_o^{-3/n}}}{(\sqrt{3} - 1)\, y_o^{-1/n} + 1} + 3^{1/4} E(\psi_o|k) - \frac{(\sqrt{3} + 1)}{2 \cdot 3^{1/4}} F(\psi_o|k)\right], \qquad (4\text{-}379)$$

wherein

$$\cos\psi_o = \frac{1 + \sqrt{3} - y_o^{1/n}}{\sqrt{3} - 1 + y_o^{1/n}}. \qquad (4\text{-}380)$$

The right-hand side of Eq. (4-379) is plotted in Fig. 10 as a function of $y_o^{1/n}$, and the fact that the curve is monotonically increasing indicates that Eq. (4-379) has only a single root for each value of α. Consequently, all values of α are allowed.

The solution of Eq. (4-1) provided by Eqs. (4-375) and (4-376) is

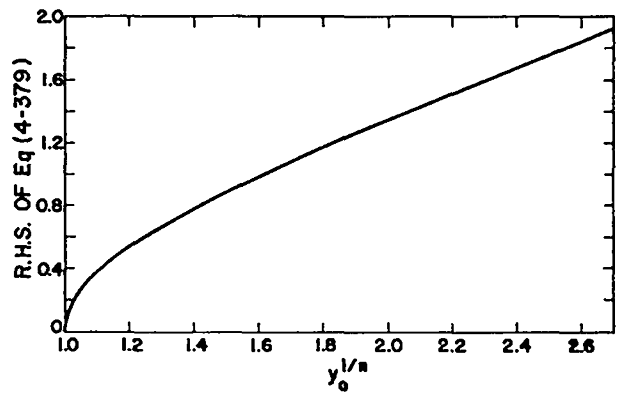

Fig. 10. Graph for determining the root of Eq. (4-379)

$$\frac{x}{a y_o^{1/n}} = \frac{(\sqrt{3} - 1)\left(\frac{y}{y_o}\right)^{1/2n}\sqrt{1 - \left(\frac{y}{y_o}\right)^{3/n}}}{(\sqrt{3} - 1)\left(\frac{y}{y_o}\right)^{1/n} + 1} + 3^{1/4} E(\psi|k) - \frac{(1 + \sqrt{3})}{2 \cdot 3^{1/4}} F(\psi|k), \qquad (4\text{-}381)$$

in which

$$\cos\psi = \frac{(1 + \sqrt{3})\left(\frac{y}{y_o}\right)^{1/n} - 1}{(\sqrt{3} - 1)\left(\frac{y}{y_o}\right)^{1/n} + 1}. \qquad (4\text{-}382)$$

The solution given in Eq. (4-381) is valid for $\lambda > -1$ and subject to the relations of Eqs. (4-369) through (4-371).

4.5.14 Results for m = 2

In the $m = 2$ case, Eq. (4-104) implies that $\nu = 0$ for all values of λ. The solution of Eq. (4-1) for the $\nu = 0$ and $\lambda = -1$ case has been discussed in Sec. 4.5.7. Accordingly, we shall now consider two further solutions of Eq. (4-1), namely, (1) that when $\nu = 0$ and $\lambda > -1$, and (2) that when $\nu = 0$ and $\lambda < -1$. Also, when $m = 2$, Eqs. (4-102) and (4-103) become

$$\frac{1}{n} = \frac{1 + \lambda}{2}, \qquad (4\text{-}383)$$

and

$$\frac{1}{a^2} = \frac{\alpha}{2}(1 + \lambda), \qquad (4\text{-}384)$$

respectively.

Now, if $\lambda > -1$, then $n > 0$ and Eq. (4-100) gives

$$\frac{1-x}{ay_o^{1/n}} = \int_{\left(\frac{1}{y_o}\right)^{1/n}}^{\left(\frac{y}{y_o}\right)^{1/n}} \frac{t}{\sqrt{1-t^2}}\,dt, \qquad (4\text{-}385)$$

so that

$$\frac{1-x}{ay_o^{1/n}} = \sqrt{1-\frac{1}{y_o^{2/n}}} - \sqrt{1-\left(\frac{y}{y_o}\right)^{2/n}}. \qquad (4\text{-}386)$$

Also, we obtain from Eq. (4-101) the relation

$$\frac{1}{ay_o^{1/n}} = \sqrt{1-\frac{1}{y_o^{2/n}}}, \qquad (4\text{-}387)$$

from which

$$y_o = \left(\frac{1+a^2}{a^2}\right)^{n/2} \qquad (4\text{-}388)$$

is the value of the solution of Eq. (4-1) at $x = 0$ for this case. Substituting Eq. (4-388) into Eq. (4-386) and simplifying produces

$$y = \left(1 + \frac{1-x^2}{a^2}\right)^{n/2}, \qquad (4\text{-}389)$$

which, in view of Eqs. (4-383) and (4-384), reduces to

$$y = \left[1 + \frac{\alpha}{2}(1+\lambda)(1-x^2)\right]^{\frac{1}{1+\lambda}}. \qquad (4\text{-}390)$$

This last relation is the solution of Eq. (4-1) when $\nu = 0$ and $\lambda > -1$ such that $y'(0) = 0$ and $y(1) = 1$, and there is no restriction of a maximum allowed value of α in it.

If we now assume that $\lambda < -1$, then $n < 0$, and the solution of Eq. (4-1) that comes out of Eq. (4-100) is

$$\frac{1-x}{ay_o^{1/n}} = \sqrt{1-\left(\frac{y}{y_o}\right)^{2/n}} - \sqrt{1-\frac{1}{y_o^{2/n}}}, \qquad (4\text{-}391)$$

and Eq. (4-101) reduces to

$$y_o^{2/n} = 1 + \frac{1}{a^2}. \qquad (4\text{-}392)$$

Because $\lambda < -1$, we write this as

$$y_o^{2/n} = 1 - \frac{\alpha}{2}(|\lambda| - 1), \qquad (4\text{-}393)$$

in view of Eq. (4-384), and, therefore, with Eq. (4-383), it is found that

$$y_o = \frac{1}{\left[1 - \frac{\alpha}{2}(|\lambda|-1)\right]^{\frac{1}{|\lambda|-1}}} \qquad (4\text{-}394)$$

is the value of the solution of Eq. (4-1) at $x = 0$ in this case. To ensure a finite, real solution at $x = 0$, Eq. (4-394) indicates that α must satisfy the inequality,

$$\alpha < \frac{2}{|\lambda| - 1}. \qquad (4\text{-}395)$$

Introducing Eq. (4-394) into Eq. (4-386) leads to the result

$$y = \frac{1}{\left[1 - \frac{\alpha}{2}(|\lambda|-1)(1-x^2)\right]^{\frac{1}{|\lambda|-1}}}, \qquad (4\text{-}396)$$

which provides the solution of Eq. (4-1) when $\nu = 0$, $\lambda < -1$, $y'(0) = 0$, $y(1) = 1$, and the inequality of Eq. (4-395) is satisfied.

4.6 Summary of Solutions of $y^\lambda y'' + \lambda y^{\lambda-1}(y')^2 + \alpha y^\nu = 0$

The 26 solutions that have been obtained for the nonlinear, two-point boundary value problem defined by Eq. (4-1) and the two boundary conditions, $y'(0) = 0$ and $y(1) = 1$, are summarized in Table IV in accordance with the value of m as defined in Eq. (4-104). In the second column of this table, the relation between λ and ν is given for the corresponding value of m. The third column gives the value of α_{max} such that a solution of Eq. (4-1) will exist. When $\lambda < -1$, then, for a given value of λ, the value of α_{max} decreases as the value of ν increases, which is a trend that is intuitively apparent. For a given value of λ such that $\lambda > -1$, the value of α_{max} increases as the value of ν decreases, which is also to be expected on the basis of physical insight. For each combination of values of λ and ν that satisfy the relation given in the second column of Table IV, the solution of Eq. (4-1), when it exists, is given by the relation whose equation number is shown in the fourth column of Table IV. Illustrative combinations of the values of λ and ν are displayed in Table V, which may be extended to the right through higher values of ν indefinitely by inspection.

Table IV. SUMMARY OF SOLUTIONS OBTAINED FOR $y^{\lambda}y'' + \lambda y^{\lambda-1}(y')^2 + \alpha y^{\nu} = 0$ WITH THE BOUNDARY CONDITIONS, $y'(0) = 0$ and $y(1) = 1$

Value of m	Relation between λ and ν	Value of α_{max}	Solution in Eq.	Solution at x = 0 in Eq.
1	$\nu = \frac{\lvert\lambda\rvert - 1}{3}$	$\alpha \leq \alpha_{max} = \frac{1.08}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-119)	(4-118)
$\frac{2}{3}$	$\nu = \frac{\lvert\lambda\rvert - 1}{2}$	$\alpha \leq \alpha_{max} = \frac{0.896}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-134)	(4-135)
$\frac{1}{2}$	$\nu = \frac{3}{5}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.817}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-139)	(4-140)
$\frac{2}{5}$	$\nu = \frac{2}{3}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.770}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-144)	(4-145)
$\frac{1}{3}$	$\nu = \frac{5}{7}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.740}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-149)	(4-150)
$\frac{2}{7}$	$\nu = \frac{3}{4}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.719}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-154)	(4-155)
0	$\nu = \lvert\lambda\rvert - 1, \lambda < -1$	$\alpha_{max} \doteq \frac{0.600}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-166)	(4-168)
0	$\nu = -(1 + \lambda), \lambda > -1$	None	(4-164)	(4-165)
$-\frac{1}{4}$	$\nu = \frac{9}{7}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.505}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-233)	(4-235)
$-\frac{2}{7}$	$\nu = \frac{4}{3}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.492}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-228)	(4-229)
$-\frac{1}{3}$	$\nu = \frac{7}{5}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.475}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-223)	(4-224)
$-\frac{2}{5}$	$\nu = \frac{3}{2}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.450}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-218)	(4-219)
$-\frac{1}{2}$	$\nu = \frac{5}{3}(\lvert\lambda\rvert - 1)$	$\alpha_{max} \doteq \frac{0.415}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-213)	(4-214)
$-\frac{2}{3}$	$\nu = 2(\lvert\lambda\rvert - 1)$	$\alpha \leq \alpha_{max} = \frac{0.352}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-208)	(4-209)
- 1	$\nu = 3(\lvert\lambda\rvert - 1)$	$\alpha \leq \alpha_{max} = \frac{0.250}{\lvert\lambda\rvert - 1}, \lambda < -1$	(4-257)	(4-252)
- 2	$\lambda = -1$ for all ν except $\nu = 0$	$\alpha \leq \alpha_{max} = \frac{0.89}{\nu}$ for $\nu > 0$	(4-268)	(4-262)
-	$\lambda = -1$ and $\nu = 0$	None	(4-280)	(4-280)
- 3	$\nu = 5(1 + \lambda)$	$\alpha \leq \alpha_{max} = \frac{0.198}{1 + \lambda}, \lambda > -1$	(4-290)	(4-292)
- 4	$\nu = 3(1 + \lambda)$	$\alpha \leq \alpha_{max} = \frac{0.354}{1 + \lambda}, \lambda > -1$	(4-304)	(4-306)
- 6	$\nu = 2(1 + \lambda)$	$\alpha \leq \alpha_{max} = \frac{0.601}{1 + \lambda}, \lambda > -1$	(4-329)	(4-322)
-	$\nu = 1 + \lambda, \lambda \neq -1$	None	(4-85)	(4-85)
6	$\nu = \frac{1 + \lambda}{2}, \lambda > -1$	None	(4-357)	(4-349)
4	$\nu = \frac{1 + \lambda}{3}, \lambda > -1$	None	(4-367)	(4-363)
3	$\nu = \frac{1 + \lambda}{5}, \lambda > -1$	None	(4-381)	(4-379)
2	$\nu = 0$ and $\lambda > -1$	None	(4-390)	(4-388)
2	$\nu = 0$ and $\lambda < -1$	$\alpha < \frac{2}{\lvert\lambda\rvert - 1}$	(4-396)	(4-394)

Table V. ILLUSTRATIVE VALUES OF λ AND ν FOR THE SOLUTIONS OF EQ. (4-1)

Value of m	ν = 1	ν = 2	ν = 3	ν = 4
1	λ = -4	λ = -7	λ = -10	λ = -13
2/3	λ = -3	λ = -5	λ = -7	λ = -9
1/2	λ = -8/3	λ = -13/3	λ = -6	λ = 23/3
2/5	λ = -5/2	λ = -4	λ = -11/2	λ = -7
1/3	λ = -12/5	λ = -19/5	λ = -26/5	λ = -33/5
2/7	λ = -7/3	λ = -11/3	λ = -5	λ = -19/3
0	λ = -2	λ = -3	λ = -4	λ = -5
-1/4	λ = -15/8	λ = -22/8	λ = -29/8	λ = -36/8
-2/7	λ = -7/4	λ = -10/4	λ = -13/4	λ = -4
-1/3	λ = -12/7	λ = -17/7	λ = -22/7	λ = -27/7
-2/5	λ = -5/3	λ = -7/3	λ = -3	λ = -11/3
-1/2	λ = -8/5	λ = -11/5	λ = -14/5	λ = -17/5
-2/3	λ = -3/2	λ = -2	λ = -5/2	λ = -3
-1	λ = -4/3	λ = -5/3	λ = -2	λ = -7/3
-2	λ = -1	λ = -1	λ = -1	λ = -1
-3	λ = -4/5	λ = -3/5	λ = -2/5	λ = -1/5
-4	λ = -2/3	λ = -1/3	λ = 0	λ = 1/3
-6	λ = -1/2	λ = 0	λ = 1/2	λ = 1
ν = 1+λ	λ = 0	λ = 1	λ = 2	λ = 3
6	λ = 1	λ = 3	λ = 5	λ = 7
4	λ = 2	λ = 5	λ = 8	λ = 11
3	λ = 4	λ = 9	λ = 14	λ = 19

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